The characteristic function for Jacobi matrices with applications

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Abstract

We introduce a class of Jacobi operators with discrete spectra which is characterized by a simple convergence condition. With any operator J from this class we associate a characteristic function as an analytic function on a suitable domain, and show that its zero set actually coincides with the set of eigenvalues of J in that domain. Further we derive sufficient conditions under which the spectrum of J is approximated by spectra of truncated finite-dimensional Jacobi matrices. As an application we construct several examples of Jacobi matrices for which the characteristic function can be expressed in terms of special functions. In more detail we study the example where the diagonal sequence of J is linear while the neighboring parallels to the diagonal are constant.

Keywords: infinite Jacobi matrix, spectral problem, characteristic function

1 Introduction

In this paper we introduce and study a class of infinite symmetric but in general complex Jacobi matrices $\mathcal{J}$ characterized by a simple convergence condition. This class is also distinguished by the discrete character of spectra of the corresponding Jacobi operators. Doing so we extend and generalize an approach to Jacobi matrices which was originally initiated, under much more restricted circumstances, in [20]. We refer to [16] for a rather general analysis of how the character of spectrum of a Jacobi operator may depend on the asymptotic behavior of weights.

For a given Jacobi matrix $\mathcal{J}$, one constructs a characteristic function $F_{\mathcal{J}}(z)$ as an analytic function on the domain $\mathbb{C}_0^{\lambda}$ obtained by excluding from the complex plane the closure of the range of the diagonal sequence λ of $\mathcal{J}$. Under some comparatively simple additional assumptions, like requiring the real part of λ to be semibounded or $\mathcal{J}$ to be real, one can show that $\mathcal{J}$ determines a unique closed Jacobi operator J on $\ell^2(\mathbb{N})$. Moreover, the spectrum of J in the domain $\mathbb{C}_0^{\lambda}$ is discrete and coincides with

the zero set of $F_{\mathcal{J}}(z)$. When establishing this relationship one may also treat the poles of $F_{\mathcal{J}}(z)$ which occur at the points from the range of the sequence λ not belonging to the set of accumulation points, however. In addition, as an important step of the proof, one makes use of an explicit formula for the Green function associated with J.

Apart of the localization of the spectrum we address too the question of approximation of the spectrum by spectra of truncated finite-dimensional Jacobi matrices. For bounded Hermitian Jacobi operators the problem has been studied, for example in [11, 3, 13]. We are aware of just a few papers, however, bringing some results in this respect also about unbounded Jacobi operators [14, 12, 18]. Our approach based on employing the characteristic function makes it possible to derive sufficient conditions under which such an approximation can be verified. This result partially reproduces and overlaps with some theorems from [12].

The characteristic function as well as numerous formulas throughout the paper are expressed in terms of a function, called $\mathfrak{F}$, defined on a subset of the space of complex sequences. In the introductory part we recall from [20] the definition of $\mathfrak{F}$ and its basic properties which are then completed by various additional facts. On the other hand, we conclude the paper with some applications of the derived results. We present several examples of Jacobi matrices for which the characteristic function can be expressed in terms of special functions (the Bessel functions or the basic hypergeometric series). A particular attention is paid to the example where the diagonal sequence λ is linear while the neighboring parallels to the diagonal are constant. In this case the characteristic equation in the variable z reads $J_{-z}(2w) = 0$, with w being a parameter, and our main concern is how the spectrum of the Jacobi operator depends on w.

2 The function $\mathfrak{F}$

2.1 Definition and basic properties

Let us recall from [20] some basic definitions and properties concerning a function $\mathfrak{F}$ defined on a subset of the linear space formed by all complex sequences $x = \{x_k\}_{k=1}^{\infty}$. Moreover, we complete this brief overview by a few additional facts.

Definition 1. Define $\mathfrak{F} : D \to \mathbb{C}$,

$$\mathfrak{F}(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \dots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \dots x_{k_m} x_{k_m+1}, \quad (1)$$

where

$$D = \left\{ \{x_k\}_{k=1}^{\infty} \subset \mathbb{C};\ \sum_{k=1}^{\infty} |x_k x_{k+1}| < \infty \right\}.$$

For a finite number of complex variables we identify $\mathfrak{F}(x_1, x_2, \dots, x_n)$ with $\mathfrak{F}(x)$ where $x = (x_1, x_2, \dots, x_n, 0, 0, 0, \dots)$. By convention, we also put $\mathfrak{F}(\emptyset) = 1$ where $\emptyset$ is the empty sequence.

Let us remark that the value of $\mathfrak{F}$ on a finite complex sequence can be expressed as the determinant of a finite Jacobi matrix. Using some basic linear algebra it is easy to show that, for $n \in \mathbb{N}$ and $\{x_j\}_{j=1}^n \subset \mathbb{C}$, one has

$$\mathfrak{F}(x_1, x_2, \ldots, x_n) = \det X_n \tag{2}$$

where

$$X_n = \begin{pmatrix} 1 & x_1 & & & & \\ x_2 & 1 & x_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & x_{n-1} & 1 & x_{n-1} \\ & & & & x_n & 1 \end{pmatrix}.$$

Note that the domain D is not a linear space. One has, however, $\ell^2(\mathbb{N}) \subset D$. In fact, the absolute value of the mth summand on the RHS of (1) is majorized by the expression

$$\sum_{\substack{k \in \mathbb{N}^m \\ k_1 < k_2 < \cdots < k_m}} |x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \ldots x_{k_m} x_{k_m+1}| \leq \frac{1}{m!} \left(\sum_{j=1}^{\infty} |x_j x_{j+1}| \right)^m.$$

Hence for $x \in D$ one has the estimate

$$|\mathfrak{F}(x)| \leq \exp\left(\sum_{k=1}^{\infty} |x_k x_{k+1}| \right). \tag{3}$$

Furthermore, $\mathfrak{F}$ satisfies the relation

$$\mathfrak{F}(x) = \mathfrak{F}(x_1, \ldots, x_k)\, \mathfrak{F}(T^k x) - \mathfrak{F}(x_1, \ldots, x_{k-1}) x_k x_{k+1} \mathfrak{F}(T^{k+1} x), \quad k = 1, 2, \ldots, \tag{4}$$

where $x \in D$ and T denotes the truncation operator from the left defined on the space of all sequences, $T(\{x_n\}_{n=1}^\infty) = \{x_{n+1}\}_{n=1}^\infty$. In particular, for $k = 1$ one gets the rule

$$\mathfrak{F}(x) = \mathfrak{F}(Tx) - x_1 x_2 \mathfrak{F}(T^2 x). \tag{5}$$

In addition, one has the symmetry property

$$\mathfrak{F}(x_1, x_2, \ldots, x_{k-1}, x_k) = \mathfrak{F}(x_k, x_{k-1}, \ldots, x_2, x_1).$$

If combined with (4), one gets

$$\mathfrak{F}(x_1, x_2, \ldots, x_{k+1}) = \mathfrak{F}(x_1, x_2, \ldots, x_k) - x_k x_{k+1}\, \mathfrak{F}(x_1, x_2, \ldots, x_{k-1}). \tag{6}$$

Lemma 2. *For $x \in D$ one has*

$$\lim_{n \to \infty} \mathfrak{F}(T^n x) = 1 \tag{7}$$

and

$$\lim_{n \to \infty} \mathfrak{F}(x_1, x_2, \ldots, x_n) = \mathfrak{F}(x). \tag{8}$$

Proof. First, similarly as in (3), one gets the estimate

$$|\mathfrak{F}(T^n x) - 1| \leq \exp\left(\sum_{k=n+1}^{\infty} |x_k x_{k+1}|\right) - 1.$$

This shows (7).

Second, in view of (4), the difference $|\mathfrak{F}(x) - \mathfrak{F}(x_1, x_2, \dots, x_n)|$ can be majorized by the expression

$$|1 - \mathfrak{F}(T^n x)| \exp\left(\sum_{k=1}^{\infty} |x_k x_{k+1}|\right) + |x_n x_{n+1}| \exp\left(2\sum_{k=1}^{\infty} |x_k x_{k+1}|\right).$$

From here one derives the (rather rough) estimate

$$|\mathfrak{F}(x) - \mathfrak{F}(x_1, x_2, \dots, x_n)| \leq 2\exp\left(2\sum_{k=1}^{\infty} |x_k x_{k+1}|\right) \sum_{k=n}^{\infty} |x_k x_{k+1}|. \tag{9}$$

This shows (8). $\square$

Proposition 3. *The function $\mathfrak{F}$ is continuous on $\ell^2(\mathbb{N})$.*

Proof. If $x \in \ell^2(\mathbb{N}) \subset D$ then from (9) one derives that, for any $n \in \mathbb{N}$,

$$|\mathfrak{F}(x) - \mathfrak{F}(x_1, x_2, \dots, x_n)| \leq 2\,\exp\big(2\,\|x\|^2\big)\,\|(I - P_{n-1})x\|^2$$

where P_m stands for the orthogonal projection on $\ell^2(\mathbb{N})$ onto the subspace spanned by the first m vectors of the standard basis. From this estimate and from the fact that $\mathfrak{F}(x_1, x_2, \dots, x_n)$ is a polynomial function the proposition readily follows. $\square$

2.2 Jacobi matrices

Let us denote by $\mathcal{J}$ an infinite Jacobi matrix of the form

$$\mathcal{J} = \begin{pmatrix} \lambda_1 & w_1 & & & \\ v_1 & \lambda_2 & w_2 & & \\ & v_2 & \lambda_3 & w_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

where $w = \{w_n\}_{n=1}^{\infty}, v = \{v_n\}_{n=1}^{\infty} \subset \mathbb{C} \setminus \{0\}$ and $\lambda = \{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{C}$. Provided any of the sequences is unbounded it is reasonable to distinguish in the notation between $\mathcal{J}$ and an operator represented by this matrix. Such an operator J need not be unique, as discussed in Subsection 3.2. Further, by J_n we denote the nth truncation of $\mathcal{J}$, i.e.

$$J_n = \begin{pmatrix} \lambda_1 & w_1 & & & \\ v_1 & \lambda_2 & w_2 & & \\ & \ddots & \ddots & \ddots & \\ & & v_{n-2} & \lambda_{n-1} & w_{n-1} \\ & & & v_{n-1} & \lambda_n \end{pmatrix}. \tag{10}$$

As is well known and in fact quite obvious, any solution $\{x_k\}$ of the formal eigenvalue equation

$$\lambda_1 x_1 + w_1 x_2 = z x_1, \quad v_{k-1} x_{k-1} + \lambda_k x_k + w_k x_{k+1} = z x_k \ \text{ for } k \geq 2, \tag{11}$$

with $z \in \mathbb{C}$, is unambiguously determined by its first component x_1. Consequently, any operator J whose matrix equals $\mathcal{J}$ may have only simple eigenvalues.

We wish to show that the characteristic function of a finite Jacobi matrix J_n can be expressed in terms of $\mathfrak{F}$. To this end, let us introduce the sequences $\{\gamma_k^{\pm}\}_{k=1}^n$ defined recursively by

$$\gamma_1^{\pm} = 1, \ \gamma_{k+1}^{+} = w_k/\gamma_k^{-} \ \text{ and } \ \gamma_{k+1}^{-} = v_k/\gamma_k^{+}, \ k \geq 1. \tag{12}$$

More explicitly, the sequence $\{\gamma_k^-\}_{k=1}^n$ can be expressed as

$$\gamma_{2k-1}^{-} = \prod_{j=1}^{k-1} \frac{v_{2j}}{w_{2j-1}}\,, \ \gamma_{2k}^{-} = v_1 \prod_{j=1}^{k-1} \frac{v_{2j+1}}{w_{2j}}\,, \ k = 1, 2, 3, \ldots.$$

As for the sequence $\{\gamma_k^+\}_{k=1}^n$, the corresponding expressions are of the same form but with w being replaced by v and vice versa. Note that if $v_k = w_k$ for all $k = 1, 2, \ldots, n-1$, then $\gamma_k^- = \gamma_k^+$ for all $k = 1, 2, \ldots, n$.

Proposition 4. *Let $\{\gamma_k^{\pm}\}_{k=1}^n$ be the sequences defined in (12). Then the equality*

$$\det(J_n - zI_n) = \left(\prod_{k=1}^{n} (\lambda_k - z)\right) \mathfrak{F}\left(\frac{\gamma_1^-\gamma_1^+}{\lambda_1 - z}, \frac{\gamma_2^-\gamma_2^+}{\lambda_2 - z}, \ldots, \frac{\gamma_n^-\gamma_n^+}{\lambda_n - z}\right) \tag{13}$$

holds for all $z \in \mathbb{C}$ (after obvious cancellations, the RHS is well defined even for $z = \lambda_k$).

Proof. Put $\tilde{\lambda}_k = \lambda_k/\gamma_k^-\gamma_k^+$. As remarked in [20, Remark 24], the Jacobi matrix J_n can be decomposed into the product $J_n = G_n^- \tilde{J}_n G_n^+$ where $G_n^{\pm} = \operatorname{diag}(\gamma_1^{\pm}, \gamma_2^{\pm}, \ldots, \gamma_n^{\pm})$ are diagonal matrices, and $\tilde{J}_n$ is a Jacobi matrix whose diagonal equals the sequence $(\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_n)$ and which has all units on the neighboring parallels to the diagonal. The proposition now readily follows from this decomposition combined with (2). □

Moreover, with the aid of (13) and using some basic calculus from linear algebra one can derive the following formula for the resolvent.

Proposition 5. *The matrix entries of the resolvent $R_n(z) = (J_n - zI_n)^{-1}$, with $z \in \mathbb{C} \setminus \operatorname{spec}(J_n)$, may be expressed as ($1 \leq i, j \leq n$)*

$$\begin{aligned} R_n(z)_{i,j} = {} & -\Omega(i,j) \left(\prod_{l=\min(i,j)}^{\max(i,j)} (z - \lambda_l)\right)^{-1} \\ & \times \mathfrak{F}\left(\left\{\frac{\gamma_l^-\gamma_l^+}{\lambda_l - z}\right\}_{l=1}^{\min(i,j)-1}\right) \mathfrak{F}\left(\left\{\frac{\gamma_l^-\gamma_l^+}{\lambda_l - z}\right\}_{l=\max(i,j)+1}^{n}\right) \mathfrak{F}\left(\left\{\frac{\gamma_l^-\gamma_l^+}{\lambda_l - z}\right\}_{l=1}^{n}\right)^{-1} \end{aligned} \tag{14}$$

where

$$\Omega(i,j) = \begin{cases} \prod_{l=i}^{j-1} w_l, & \text{if } i < j, \\ 1, & \text{if } i = j, \\ \prod_{l=j}^{i-1} v_l, & \text{if } i > j. \end{cases}$$

In the remainder of the paper we concentrate, however, on symmetric Jacobi matrices with $v = w$, i.e. we put

$$\mathcal{J} = \begin{pmatrix} \lambda_1 & w_1 & & & \\ w_1 & \lambda_2 & w_2 & & \\ & w_2 & \lambda_3 & w_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

where $\lambda = \{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{C}$ and $w = \{w_n\}_{n=1}^{\infty} \subset \mathbb{C} \setminus \{0\}$. In that case some definitions introduced above simplify. First of all, one has $\gamma_k^- = \gamma_k^+ = \gamma_k$ where

$$\gamma_{2k-1} = \prod_{j=1}^{k-1} \frac{w_{2j}}{w_{2j-1}}\,,\ \gamma_{2k} = w_1 \prod_{j=1}^{k-1} \frac{w_{2j+1}}{w_{2j}}\,,\ k = 1, 2, 3, \ldots .$$

Then $\gamma_k \gamma_{k+1} = w_k$.

2.3 More on the function $\mathfrak{F}$

In [20] one can find two examples of special functions expressed in terms of $\mathfrak{F}$. The first example is concerned with the Bessel functions of the first kind. In more detail, for $w, \nu \in \mathbb{C}$, $\nu \notin -\mathbb{N}$, one has

$$J_\nu(2w) = \frac{w^\nu}{\Gamma(\nu+1)}\, \mathfrak{F}\left(\left\{ \frac{w}{\nu+k} \right\}_{k=1}^{\infty} \right). \tag{15}$$

Notice that jointly with (3) this implies

$$|J_\nu(2w)| \le \left| \frac{w^\nu}{\Gamma(\nu+1)} \right| \exp\left(\sum_{k=1}^{\infty} \left| \frac{w^2}{(\nu+k)(\nu+k+1)} \right| \right). \tag{16}$$

In the second example one shows that the formula

$$\mathfrak{F}\big(\{t^{k-1}w\}_{k=1}^{\infty}\big) = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{t^{m(2m-1)} w^{2m}}{(1-t^2)(1-t^4)\ldots(1-t^{2m})} = {}_0\phi_1(;0;t^2,-tw^2) \tag{17}$$

holds for $t, w \in \mathbb{C}$, $|t| < 1$. Here ${}_0\phi_1$ is the basic hypergeometric series (also called q-hypergeometric series) being defined by

$${}_0\phi_1(;b;q,z) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)}}{(q;q)_k (b;q)_k}\, z^k,$$

and

$$(a;q)_k = \prod_{j=0}^{k-1} \left(1 - aq^j\right), \ k = 0, 1, 2, \ldots,$$

is the q-Pochhammer symbol; see [9] for more details.

In this connection let us recall one more identity proved in [20, Lemma 9], namely

$$\begin{aligned} &u_1 \mathfrak{F}(u_2, u_3, \ldots, u_n) \mathfrak{F}(v_1, v_2, \ldots, v_n) - v_1 \mathfrak{F}(u_1, u_2, \ldots, u_n) \mathfrak{F}(v_2, v_3, \ldots, v_n) \\ &= \sum_{j=1}^{n} \left(\prod_{k=1}^{j-1} u_k v_k \right) (u_j - v_j) \mathfrak{F}(u_{j+1}, u_{j+2}, \ldots, u_n) \mathfrak{F}(v_{j+1}, v_{j+2}, \ldots, v_n). \end{aligned}$$

For the particular choice

$$u_k = \frac{w}{\mu + k}, \ v_k = \frac{w}{\nu + k}, \ 1 \leq k \leq n,$$

one can consider the limit $n \to \infty$. Using (15) and (16) one arrives at the equation

$$J_\mu(2w) J_{\nu+1}(2w) - J_{\mu+1}(2w) J_\nu(2w) = \frac{\mu - \nu}{w} \sum_{j=1}^{\infty} J_{\mu+j}(2w) J_{\nu+j}(2w). \tag{18}$$

Definition (1) can naturally be extended to more general ranges of indices. For any sequence $\{x_n\}_{n=N_1}^{N_2}$, $N_1, N_2 \in \mathbb{Z} \cup \{-\infty, +\infty\}$, $N_1 \leq N_2 + 1$, (if $N_1 = N_2 + 1 \in \mathbb{Z}$ then the sequence is considered as empty) such that

$$\sum_{k=N_1}^{N_2-1} |x_k x_{k+1}| < \infty$$

one can define

$$\mathfrak{F}\Big(\{x_k\}_{k=N_1}^{N_2}\Big) = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k \in \mathcal{I}(N_1,N_2,m)} x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \ldots x_{k_m} x_{k_m+1}$$

where

$$\mathcal{I}(N_1, N_2, m) = \{k \in \mathbb{Z}^m; \ k_j + 2 \leq k_{j+1} \text{ for } 1 \leq j \leq m-1, \ N_1 \leq k_1, \ k_m < N_2\}.$$

With this definition one can generalize the rule (4). Now one has

$$\mathfrak{F}\Big(\{x_k\}_{k=N_1}^{N_2}\Big) = \mathfrak{F}\big(\{x_k\}_{k=N_1}^{n}\big) \mathfrak{F}\Big(\{x_k\}_{k=n+1}^{N_2}\Big) - x_n x_{n+1} \mathfrak{F}\big(\{x_k\}_{k=N_1}^{n-1}\big) \mathfrak{F}\Big(\{x_k\}_{k=n+2}^{N_2}\Big) \tag{19}$$

provided $n \in \mathbb{Z}$ satisfies $N_1 \leq n < N_2$.

This extension also opens the way for applications of the function $\mathfrak{F}$ to bilateral difference equations. Suppose that sequences $\{w_n\}_{n=-\infty}^{\infty}$ and $\{\zeta_n\}_{n=-\infty}^{\infty}$ are such that $w_n \neq 0$, $\zeta_n \neq 0$ for all n and

$$\sum_{k=-\infty}^{\infty}\left|\frac{w_k^2}{\zeta_k\zeta_{k+1}}\right|<\infty.$$

Consider the difference equation

$$w_n u_{n+1}-\zeta_n u_n+w_{n-1}u_{n-1}=0,\ n\in\mathbb{Z}. \tag{20}$$

Define the sequence $\{\mathcal{P}_n\}_{n\in\mathbb{Z}}$ by $\mathcal{P}_0=1$ and $\mathcal{P}_{n+1}=(w_n/\zeta_{n+1})\mathcal{P}_n$ for all n. Hence

$$\mathcal{P}_n=\prod_{k=1}^{n}\frac{w_{k-1}}{\zeta_k}\ \text{ for } n>0,\ \mathcal{P}_0=1,\ \mathcal{P}_n=\prod_{k=n+1}^{0}\frac{\zeta_k}{w_{k-1}}\ \text{ for } n<0.$$

The sequence $\{\gamma_n\}_{n\in\mathbb{Z}}$ is again defined so that $\gamma_1=1$ and $\gamma_n\gamma_{n+1}=w_n$ for all $n\in\mathbb{Z}$. Hence

$$\gamma_{2k-1}=\prod_{j=1}^{k-1}\frac{w_{2j}}{w_{2j-1}}\,,\ \gamma_{2k}=w_1\prod_{j=1}^{k-1}\frac{w_{2j+1}}{w_{2j}}\,,\ \text{for } k=1,2,3,\ldots,$$

and

$$\gamma_{2k-1}=\prod_{j=k}^{0}\frac{w_{2j-1}}{w_{2j}}\,,\ \gamma_{2k}=w_1\prod_{j=k}^{0}\frac{w_{2j}}{w_{2j+1}}\,,\ \text{for } k=0,-1,-2,\ldots.$$

Then the sequences $\{f_n\}_{n\in\mathbb{Z}}$ and $\{g_n\}_{n\in\mathbb{Z}}$,

$$f_n=\mathcal{P}_n\,\mathfrak{F}\!\left(\left\{\frac{\gamma_k^2}{\zeta_k}\right\}_{k=n+1}^{\infty}\right),\ g_n=\frac{1}{w_{n-1}\mathcal{P}_{n-1}}\,\mathfrak{F}\!\left(\left\{\frac{\gamma_k^2}{\zeta_k}\right\}_{k=-\infty}^{n-1}\right), \tag{21}$$

represent two solutions of the bilateral difference equation (20).

For two solutions $u=\{u_n\}_{n\in\mathbb{Z}}$ and $v=\{v_n\}_{n\in\mathbb{Z}}$ of (20) the Wronskian is introduced as

$$\mathcal{W}(u,v)=w_n\left(u_n v_{n+1}-u_{n+1}v_n\right).$$

As is well known, this is a constant independent of the index n. Moreover, two solutions are linearly dependent iff their Wronskian vanishes. For the solutions f and g given in (21) one can use (19) to evaluate their Wronskian getting

$$\mathcal{W}(f,g)=\mathfrak{F}\!\left(\left\{\frac{\gamma_n^2}{\zeta_n}\right\}_{n=-\infty}^{\infty}\right).$$

One may also consider an application of a discrete analogue of Green's formula to the solutions (21) [2]. In general, suppose that sequences $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ solve respectively the difference equations

$$w_n u_{n+1}-\zeta_n^{(1)}u_n+w_{n-1}u_{n-1}=0,\ w_n v_{n+1}-\zeta_n^{(2)}v_n+w_{n-1}v_{n-1}=0,\ n\in\mathbb{N}. \tag{22}$$

In that case it is well known and easy to check that

$$\sum_{j=1}^{n}\left(\zeta_j^{(1)}-\zeta_j^{(2)}\right)u_j v_j=w_0\left(u_0v_1-u_1v_0\right)-w_n\left(u_n v_{n+1}-u_{n+1}v_n\right). \tag{23}$$

Proposition 6. *Suppose that the convergence condition*

$$\sum_{k=1}^{\infty}\left|\frac{w_k^2}{\zeta_k\zeta_{k+1}}\right|<\infty$$

is satisfied for the both difference equations in (22). Moreover, assume that

$$\sup_{n\geq 1}\left|\frac{w_n^2}{\zeta_n^{(1)}\zeta_{n+1}^{(2)}}\right|<\infty \quad and \quad \sup_{n\geq 1}\left|\frac{w_n^2}{\zeta_n^{(2)}\zeta_{n+1}^{(1)}}\right|<\infty.$$

Then the corresponding solutions $f^{(1)}$, $f^{(2)}$ from (21) fulfill

$$\sum_{j=1}^{\infty}\left(\zeta_j^{(1)}-\zeta_j^{(2)}\right)f_j^{(1)}f_j^{(2)}=w_0\left(f_0^{(1)}f_1^{(2)}-f_1^{(1)}f_0^{(2)}\right). \tag{24}$$

Proof. In view of (23) it suffices to show that

$$\lim_{n\to\infty}w_nf_n^{(1)}f_{n+1}^{(2)}=\lim_{n\to\infty}w_nf_{n+1}^{(1)}f_n^{(2)}=0. \tag{25}$$

By the convergence assumption, for all $n>n_0$ one has

$$|w_n|\leq\frac{1}{2}\sqrt{|\zeta_n^{(1)}||\zeta_{n+1}^{(1)}|},\ |w_n|\leq\frac{1}{2}\sqrt{|\zeta_n^{(2)}||\zeta_{n+1}^{(2)}|}.$$

Using (3), after some straightforward manipulations one gets the estimate

$$\begin{aligned}\left|w_nf_n^{(1)}f_{n+1}^{(2)}\right| \;&\leq\; 2^{-2(n-n_0)}\exp\left(\sum_{k=1}^{\infty}\left|\frac{w_k^2}{\zeta_k^{(1)}\zeta_{k+1}^{(1)}}\right|+\left|\frac{w_k^2}{\zeta_k^{(2)}\zeta_{k+1}^{(2)}}\right|\right)\prod_{k=1}^{n_0}\left|\frac{w_{k-1}^2}{\zeta_k^{(1)}\zeta_k^{(2)}}\right|\\ &\quad\times|\zeta_{n_0}^{(1)}\zeta_{n_0}^{(2)}|^{1/2}\ \frac{|w_n|}{\left|\zeta_n^{(1)}\zeta_{n+1}^{(2)}\right|^{1/2}}.\end{aligned}$$

This implies (25). ∎

In the literature on Jacobi matrices one encounters a construction of an infinite matrix associated with the bilateral difference equation (20) [21, § 1.1], [10, Theorem 1.2]. Let us define the matrix $\mathfrak{J}$ with entries $\mathfrak{J}(m,n)$, $m,n\in\mathbb{Z}$, so that for every fixed m, the sequence $u_n=\mathfrak{J}(m,n)$, $n\in\mathbb{Z}$, solves (20) with the initial conditions $\mathfrak{J}(m,m)=0$, $\mathfrak{J}(m,m+1)=1/w_m$.

Using (6) one verifies that, for $m<n$,

$$\mathfrak{J}(m,n)=\frac{1}{w_m}\left(\prod_{j=m+1}^{n-1}\frac{\zeta_j}{w_j}\right)\mathfrak{F}\left(\frac{\gamma_{m+1}^2}{\zeta_{m+1}},\frac{\gamma_{m+2}^2}{\zeta_{m+2}},\ldots,\frac{\gamma_{n-1}^2}{\zeta_{n-1}}\right).$$

Moreover, it is quite obvious that, for all $m,n\in\mathbb{Z}$,

$$\mathfrak{J}(m,n)=\frac{1}{\mathcal{W}(u,v)}\left(u_mv_n-v_mu_n\right),$$

where $\{u_n\}$, $\{v_n\}$ is any couple of independent solutions of (20). Hence the matrix $\mathfrak{J}$ is antisymmetric. It also follows that, $\forall m, n, k, \ell \in \mathbb{Z}$,

$$\mathfrak{J}(m,k)\mathfrak{J}(n,\ell) - \mathfrak{J}(m,\ell)\mathfrak{J}(n,k) = \mathfrak{J}(m,n)\mathfrak{J}(k,\ell).$$

Example 7. As an example let us again have a look at the particular case where $w_n = w$, $\zeta_n = \nu + n$ for all $n \in \mathbb{Z}$ and some $w, \nu \in \mathbb{C}$, $w \neq 0$, $\nu \notin \mathbb{Z}$. One finds, with the aid of (15), that the solutions (21) now read

$$f_n = \Gamma(\nu+1)\, w^{-\nu} J_{\nu+n}(2w),\ g_n = \frac{(-1)^n \pi}{\sin(\pi\nu)\Gamma(\nu+1)}\, w^{\nu} J_{-\nu-n}(2w).$$

Hence the Wronskian equals

$$\mathcal{W}(f,g) = \frac{\pi w}{\sin(\pi\nu)}\left(-J_\nu(2w)J_{-\nu-1}(2w) - J_{\nu+1}(2w)J_{-\nu}(2w)\right) = \mathfrak{F}\left(\left\{\frac{w}{\nu+n}\right\}_{n=-\infty}^{\infty}\right).$$

Recalling once more (15) we note that the RHS equals

$$\lim_{N\to\infty} \mathfrak{F}\left(\left\{\frac{w}{\nu-N+n}\right\}_{n=1}^{\infty}\right) = \lim_{N\to\infty} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\, \frac{\Gamma(\nu-N+1)}{\Gamma(\nu-N+n+1)}\, w^{2n} = 1.$$

Thus one gets the well known relation [1, Eq. 9.1.15]

$$J_{\nu+1}(2w)J_{-\nu}(2w) + J_\nu(2w)J_{-\nu-1}(2w) = -\frac{\sin(\pi\nu)}{\pi w}\,. \tag{26}$$

Concerning the matrix $\mathfrak{J}$, this particular choice brings us to the case discussed in [20, Proposition 22]. Then the Bessel functions $Y_{n+\nu}(2w)$ and $J_{n+\nu}(2w)$, depending on the index $n \in \mathbb{Z}$, represent other two linearly independent solutions of (20). Since [1, Eq. 9.1.16]

$$J_{\nu+1}(z)Y_\nu(z) - J_\nu(z)Y_{\nu+1}(z) = \frac{2}{\pi z}$$

one finds that

$$\mathfrak{J}(m,n) = \pi\left(Y_{m+\nu}(2w)J_{n+\nu}(2w) - J_{m+\nu}(2w)Y_{n+\nu}(2w)\right).$$

Moreover, for $\sigma = m + \mu$ and $k = n - m > 0$ one has

$$J_{\sigma+k}(2w)Y_\sigma(2w) - J_\sigma(2w)Y_{\sigma+k}(2w) = \frac{\Gamma(\sigma+k)}{\pi w^k\,\Gamma(\sigma+1)}\, \mathfrak{F}\left(\left\{\frac{w}{\sigma+j}\right\}_{j=1}^{k-1}\right).$$

Finally, putting $\zeta_n^{(1)} = \mu + n$, $\zeta_n^{(2)} = \nu + n$ and $w_n = w$, $\forall n \in \mathbb{N}$, in equation (22), one verifies that (24) holds true and reveals this way once more the identity (18).

3 A class of Jacobi operators with point spectra

3.1 The characteristic function

Being inspired by Proposition 4 and notably by equation (13), we introduce the (renormalized) characteristic function associated with a Jacobi matrix $\mathcal{J}$ as

$$F_{\mathcal{J}}(z) := \mathfrak{F}\left(\left\{\frac{\gamma_n^2}{\lambda_n - z}\right\}_{n=1}^{\infty}\right). \tag{27}$$

It is treated as a complex function of a complex variable z and is well defined provided the sequence in the argument of $\mathfrak{F}$ belongs to the domain D. Let us show that this is guaranteed under the assumption that there exists $z_0 \in \mathbb{C}$ such that

$$\sum_{n=1}^{\infty} \left| \frac{w_n^2}{(\lambda_n - z_0)(\lambda_{n+1} - z_0)} \right| < \infty. \tag{28}$$

For $\lambda = \{\lambda_n\}_{n=1}^{\infty}$ let us denote

$$\mathbb{C}_0^{\lambda} := \mathbb{C} \setminus \overline{\{\lambda_n;\, n \in \mathbb{N}\}}.$$

Clearly,

$$\overline{\{\lambda_n;\, n \in \mathbb{N}\}} = \{\lambda_n;\, n \in \mathbb{N}\} \cup \operatorname{der}(\lambda)$$

where $\operatorname{der}(\lambda)$ stands for the set of all finite accumulation points of the sequence λ (i.e., $\operatorname{der}(\lambda)$ is equal to the set of limit points of all possible convergent subsequences of λ).

Lemma 8. *Let condition (28) be fulfilled for at least one $z_0 \in \mathbb{C}_0^{\lambda}$. Then the series*

$$\sum_{n=1}^{\infty} \frac{w_n^2}{(\lambda_n - z)(\lambda_{n+1} - z)} \tag{29}$$

converges absolutely and locally uniformly in z on $\mathbb{C}_0^{\lambda}$. Moreover,

$$\forall z \in \mathbb{C}_0^{\lambda}, \quad \lim_{n\to\infty} \mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - z}\right\}_{k=1}^{n}\right) = F_{\mathcal{J}}(z), \tag{30}$$

and the convergence is locally uniform on $\mathbb{C}_0^{\lambda}$. Consequently, $F_{\mathcal{J}}(z)$ is a well defined analytic function on $\mathbb{C}_0^{\lambda}$.

Proof. Let $K \subset \mathbb{C}_0^{\lambda}$ be a compact subset. Then the ratio

$$\frac{|\lambda_n - z_0|}{|\lambda_n - z|} \leq 1 + \frac{|z - z_0|}{|\lambda_n - z|}$$

admits an upper bound, uniform in $z \in K$ and $n \in \mathbb{N}$. The uniform convergence on K of the series (29) thus becomes obvious.

The limit (30) follows from Lemma 2. Moreover, using (30) and also (6), (3) one has

$$\left|\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - z}\right\}_{k=1}^{n}\right) - F_{\mathcal{J}}(z)\right| \leq \sum_{l=n}^{\infty}\left|\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - z}\right\}_{k=1}^{l}\right) - \mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - z}\right\}_{k=1}^{l+1}\right)\right|$$
$$\leq \sum_{l=n}^{\infty}\left|\frac{w_l^2}{(\lambda_l - z)(\lambda_{l+1} - z)}\right| \exp\left(\sum_{k=1}^{\infty}\left|\frac{w_k^2}{(\lambda_k - z)(\lambda_{k+1} - z)}\right|\right).$$

From this estimate and the locally uniform convergence of the series (29) one deduces the locally uniform convergence of the sequence of functions (30). □

By a closer inspection one finds that, under the assumptions of Lemma 8, the function $F_{\mathcal{J}}(z)$ is meromorphic on $\mathbb{C} \setminus \operatorname{der}(\lambda)$ with poles at the points $z = \lambda_n$ for some $n \in \mathbb{N}$ (not belonging to $\operatorname{der}(\lambda)$, however). For any such z, the order of the pole is less than or equal to $r(z)$ where

$$r(z) := \sum_{k=1}^{\infty} \delta_{z,\lambda_k}$$

is the number of members of the sequence λ coinciding with z (hence $r(z) = 0$ for $z \in \mathbb{C}_0^{\lambda}$). To see this, suppose that $r(z) \geq 1$ and let M be the maximal index such that $\lambda_M = z$. Using (4) one derives that, for $u \in \mathbb{C}_0^{\lambda}$,

$$\begin{aligned} F_{\mathcal{J}}(u) \;=\; & \mathfrak{F}\left(\left\{\frac{\gamma_n^2}{\lambda_n - u}\right\}_{n=1}^{M}\right)\mathfrak{F}\left(\left\{\frac{\gamma_n^2}{\lambda_n - u}\right\}_{n=M+1}^{\infty}\right) \\ & + \mathfrak{F}\left(\left\{\frac{\gamma_n^2}{\lambda_n - u}\right\}_{n=1}^{M-1}\right)\frac{\gamma_M^2\gamma_{M+1}^2}{(u - z)(\lambda_{M+1} - u)}\,\mathfrak{F}\left(\left\{\frac{\gamma_n^2}{\lambda_n - u}\right\}_{n=M+2}^{\infty}\right). \end{aligned}$$

The RHS clearly has a pole at the point $u = z$ of order at most $r(z)$.

3.2 The Jacobi operator J

Our goal is to investigate spectral properties of a closed operator J on $\ell^2(\mathbb{N})$ whose matrix in the standard basis coincides with $\mathcal{J}$. Provided the Jacobi matrix does not determine a bounded operator, however, there need not be a unique way how to introduce J. But among all admissible operators one may distinguish two particular cases which may respectively be regarded, in a natural way, as the minimal and the maximal operator with the required properties; see, for instance, [4].

Definition 9. The operator $J_{\max}$ is defined so that

$$\operatorname{Dom}(J_{\max}) = \{y \in \ell^2(\mathbb{N});\ \mathcal{J}y \in \ell^2(\mathbb{N})\},$$

and one sets $J_{\max}y = \mathcal{J}y$, $\forall y \in \operatorname{Dom} J_{\max}$. Here and in what follows $\mathcal{J}y$ is understood as the formal matrix product while treating y as a column vector. To define the operator $J_{\min}$ one first introduces the operator $\dot{J}$ so that $\operatorname{Dom}(\dot{J})$ is the linear hull of the standard basis, and again $\dot{J}y = \mathcal{J}y$ for all $y \in \operatorname{Dom}(\dot{J})$. $\dot{J}$ is known to be closable [4], and $J_{\min}$ is defined as the closure of $\dot{J}$.

One has the following relations between the operators $J_{\min}$, $J_{\max}$ and their adjoint operators [4, Lemma 2.1]. Let $\mathcal{J}^H$ designates the Jacobi matrix obtained from $\mathcal{J}$ by taking the complex conjugate of each entry. Then $J_{\min}^* = J_{\max}^H$, $J_{\max}^* = J_{\min}^H$. In particular, the maximal operator $J_{\max}$ is a closed extension of $J_{\min}$. It is even true that any closed operator J whose domain contains the standard basis and whose matrix in this basis equals $\mathcal{J}$ fulfills $J_{\min} \subset J \subset J_{\max}$. Moreover, if $\mathcal{J}$ is Hermitian, i.e. $\mathcal{J} = \mathcal{J}^H$ (which means nothing but $\mathcal{J}$ is real), then $J_{\min}^* = J_{\max} \supset J_{\min}$. Hence $J_{\min}$ is symmetric with the deficiency indices either $(0,0)$ or $(1,1)$.

We are primarily interested in the situation where $J_{\min} = J_{\max}$ since then there exists a unique closed operator J defined by the Jacobi matrix $\mathcal{J}$, and it turns out that the spectrum of J is determined in a well defined sense by the characteristic function $F_{\mathcal{J}}(z)$. If this happens $\mathcal{J}$ is sometimes called proper [4].

Let us recall more details on this property. We remind the reader that the orthogonal polynomials of the first kind, $p_n(z)$, are defined by the recurrence

$$w_{n-1}p_{n-1}(z) + \lambda_n p_n(z) + w_n p_{n+1}(z) = z\, p_n(z), \quad n = 1, 2, 3, \ldots,$$

with the initial conditions $p_0(z) = 1$, $p_1(z) = (z - \lambda_1)/w_1$. The orthogonal polynomials of the second kind, $q_n(z)$, obey the same recurrence but the initial conditions are $q_0(z) = 0$, $q_1(z) = 1/w_1$; see [2, 5]. It is not difficult to verify that these polynomials are expressible in terms of the function $\mathfrak{F}$ as follows:

$$p_n(z) = \left(\prod_{k=1}^{n} \frac{z - \lambda_k}{w_k} \right) \mathfrak{F}\left(\left\{ \frac{\gamma_l^2}{\lambda_l - z} \right\}_{l=1}^{n} \right), \quad n = 0, 1, 2 \ldots,$$

and

$$q_n(z) = \frac{1}{w_1} \left(\prod_{k=2}^{n} \frac{z - \lambda_k}{w_k} \right) \mathfrak{F}\left(\left\{ \frac{\gamma_l^2}{\lambda_l - z} \right\}_{l=2}^{n} \right), \quad n = 1, 2, 3 \ldots.$$

The complex Jacobi matrix $\mathcal{J}$ is called determinate if at least one of the sequences $p(0) = \{p_n(0)\}_{n=0}^{\infty}$ or $q(0) = \{q_n(0)\}_{n=0}^{\infty}$ is not an element of $\ell^2(\mathbb{Z}_+)$. For real Jacobi matrices there exits a parallel terminology. Instead of determinate one calls $\mathcal{J}$ limit point at $+\infty$, and instead of indeterminate one calls $\mathcal{J}$ limit circle at $+\infty$, see [21, p. 48]. According to [22, Theorem 22.1], $\mathcal{J}$ is indeterminate if both $p(z)$ and $q(z)$ are elements of ℓ^2 for at least one $z \in \mathbb{C}$, and in this case they are elements of ℓ^2 for all $z \in \mathbb{C}$. For a real Jacobi matrix $\mathcal{J}$ one can prove that it is proper if and only if it is determinate (or, in another terminology, limit point), see [2, pp. 138-141] or [21, Lemma 2.16].

For complex Jacobi matrices one can also specify assumptions under which $J_{\min} = J_{\max}$. In what follows, $\rho(A)$ designates the resolvent set of a closed operator A. Concerning the essential spectrum, one observes that $\mathrm{spec}_{ess}(J_{\min}) = \mathrm{spec}_{ess}(J_{\max})$ [4, Eq. 2.10]. Hence if $\rho(J_{\max}) \neq \emptyset$ then $\mathrm{spec}_{ess}(J_{\min}) \neq \mathbb{C}$. Moreover, in that case $\mathcal{J}$ is determinate [4, Theorem 2.11 (a)] and proper [4, Theorem 2.6 (a)]. This way one extracts from [4] the following result.

Theorem 10. *If $\rho(J_{max}) \neq \emptyset$ then $J_{min} = J_{max}$.*

3.3 The spectrum and the zero set of the characteristic function

Let us define

$$\mathfrak{Z}(\mathcal{J}) := \left\{ z \in \mathbb{C} \setminus \operatorname{der}(\lambda);\ \lim_{u \to z} (u-z)^{r(z)} F_{\mathcal{J}}(u) = 0 \right\}. \tag{31}$$

Of course, $\mathfrak{Z}(\mathcal{J}) \cap \mathbb{C}_0^{\lambda}$ is nothing but the set of zeros of $F_{\mathcal{J}}(z)$. Further, for $k \in \mathbb{Z}_+$ and $z \in \mathbb{C} \setminus \operatorname{der}(\lambda)$ we put

$$\xi_k(z) := \lim_{u \to z} (u-z)^{r(z)} \left(\prod_{l=1}^{k} \frac{w_{l-1}}{u - \lambda_l} \right) \mathfrak{F}\left(\left\{ \frac{\gamma_l^2}{\lambda_l - u} \right\}_{l=k+1}^{\infty} \right), \tag{32}$$

where one sets $w_0 = 1$. One observes that for $k \geq M$, where $M = M_z$ is either the maximal index, if any, such that $z = \lambda_M$, or $M = 0$ otherwise,

$$\xi_k(z) = \prod_{l=1}^{k} w_{l-1} \left(\prod_{\substack{l=1 \\ \lambda_l \neq z}}^{k} (z - \lambda_l) \right)^{-1} \mathfrak{F}\left(\left\{ \frac{\gamma_l^2}{\lambda_l - z} \right\}_{l=k+1}^{\infty} \right). \tag{33}$$

Proposition 11. *Let condition (28) be fulfilled for at least one $z_0 \in \mathbb{C}_0^{\lambda}$. If*

$$\xi_0(z) \equiv \lim_{u \to z} (u-z)^{r(z)} F_{\mathcal{J}}(u) = 0$$

for some $z \in \mathbb{C} \setminus \operatorname{der}(\lambda)$, then z is an eigenvalue of J_{max} and

$$\xi(z) := (\xi_1(z), \xi_2(z), \xi_3(z), \ldots)$$

is the corresponding eigenvector.

Proof. Using (5) one verifies that if $\xi_0(z) = 0$ then the column vector $\xi(z)$ solves the matrix equation $\mathcal{J}\xi(z) = z\xi(z)$. To complete the proof one has to show that $\xi(z)$ does not vanish and belongs to $\ell^2(\mathbb{N})$.

First, we claim that $\xi_1(z) \neq 0$. Suppose, on the contrary, that $\xi_1(z) = 0$. Then the formal eigenvalue equation (which is a second order recurrence) implies $\xi(z) = 0$. From (33) it follows that

$$\mathfrak{F}\left(\left\{ \frac{\gamma_l^2}{\lambda_l - z} \right\}_{l=k+1}^{\infty} \right) = 0$$

for all $k \geq M = M_z$. This equality is in contradiction with (7), however.

Second, suppose $z \notin \operatorname{der}(\lambda)$ is fixed. By Lemma 8, there exists $N \in \mathbb{N}$, $N > M$, such that

$$|w_n^2| \leq |\lambda_n - z||\lambda_{n+1} - z|/2,\ \forall n \geq N.$$

Let us denote

$$C = \prod_{l=1}^{N} |w_{l-1}|^2 \prod_{\substack{l=1 \\ \lambda_l \neq z}}^{N} |z - \lambda_l|^{-2}.$$

Using also (3) one can estimate

$$\begin{aligned}\sum_{k=N}^{\infty}|\xi_k(z)|^2 \;&=\; \sum_{k=N}^{\infty}\prod_{l=1}^{k}|w_{l-1}|^2\prod_{\substack{l=1\\ \lambda_l\neq z}}^{k}|z-\lambda_l|^{-2}\left|\mathfrak{F}\left(\left\{\frac{\gamma_l^2}{\lambda_l-z}\right\}_{l=k+1}^{\infty}\right)\right|^2\\ &\leq C\exp\left(2\sum_{k=N+1}^{\infty}\left|\frac{w_k^2}{(\lambda_k-z)(\lambda_{k+1}-z)}\right|\right)\sum_{k=N}^{\infty}\prod_{l=N+1}^{k}\left(\frac{1}{2}\left|\frac{z-\lambda_{l-1}}{z-\lambda_l}\right|\right).\end{aligned}$$

Since $|\lambda_k - z| \geq \tau$ for all $k > M$ and some $\tau > 0$, the RHS is finite. □

Further we wish to prove a statement converse to Proposition 11. Our approach is based on a formula for the Green function generalizing a similar result known for the finite-dimensional case; see (14).

Proposition 12. *Let condition (28) be fulfilled for at least one $z_0 \in \mathbb{C}_0^{\lambda}$. If $z \in \mathbb{C}\setminus \operatorname{der}(\lambda)$ does not belong to the zero set $\mathfrak{Z}(\mathcal{J})$ then $z \in \rho(J_{max})$ and the Green function for the spectral parameter z,*

$$G(z;i,j) := \langle e_i, (J_{max}-z)^{-1}e_j\rangle,\ \ i,j\in\mathbb{N},$$

(a matrix in the standard basis) is given by the formula

$$\begin{aligned}G(z;i,j) \;&=\; -\frac{1}{w_{\max(i,j)}}\left(\prod_{l=\min(i,j)}^{\max(i,j)}\frac{w_l}{z-\lambda_l}\right) \qquad (34)\\ &\times\mathfrak{F}\left(\left\{\frac{\gamma_l^2}{\lambda_l-z}\right\}_{l=1}^{\min(i,j)-1}\right)\mathfrak{F}\left(\left\{\frac{\gamma_l^2}{\lambda_l-z}\right\}_{l=\max(i,j)+1}^{\infty}\right)\mathfrak{F}\left(\left\{\frac{\gamma_l^2}{\lambda_l-z}\right\}_{l=1}^{\infty}\right)^{-1}.\end{aligned}$$

In particular, for the Weyl m-function one has

$$m(z) := G(z;1,1) = \frac{1}{\lambda_1-z}\,\mathfrak{F}\left(\left\{\frac{\gamma_l^2}{\lambda_l-z}\right\}_{l=2}^{\infty}\right)\mathfrak{F}\left(\left\{\frac{\gamma_l^2}{\lambda_l-z}\right\}_{l=1}^{\infty}\right)^{-1}. \qquad (35)$$

If, in addition, $|\lambda_n| \to \infty$ as $n \to \infty$ then $\operatorname{der}(\lambda) = \emptyset$ *and for every $z \in \mathbb{C}\setminus\mathfrak{Z}(\mathcal{J})$, the resolvent $(J_{max}-z)^{-1}$ is compact.*

Proof. Denote by $R(z)_{i,j}$ the RHS of (34). Thus $R(z)$ is an infinite matrix provided its entries $R(z)_{i,j}$, $i,j \in \mathbb{N}$, make good sense. Suppose that a complex number z does not belong to $\mathfrak{Z}(J)\cup\operatorname{der}(\lambda)$. By Lemma 8, in that case the RHS of (34) is well defined. By inspection of the expression one finds that this is so even if z happens to coincide with a member λ_k of the sequence λ not belonging to $\operatorname{der}(\lambda)$, i.e. the seeming singularity at $z = \lambda_k$ is removable. For the sake of simplicity we assume in the remainder of the proof, however, that z does not belong to the range of the sequence λ. The only purpose of this assumption is just to simplify the discussion and to avoid more complex expressions but otherwise it is not essential for the result.

First let us show that there exists a constant C, possibly depending on z but independent of the indices i, j, such that

$$|R(z)_{i,j}| \leq C\,2^{-|i-j|},\ \forall i,j \in \mathbb{N}. \tag{36}$$

To this end, denote

$$\tau = \inf\{|z-\lambda_n|;\ n \in \mathbb{N}\} > 0.$$

Assuming (28), one can choose $n_0 \in \mathbb{N}$ so that, for all $n \geq n_0$,

$$|w_n|^2 \leq |\lambda_n - z|\,|\lambda_{n+1} - z|/4. \tag{37}$$

Let us assume, for the sake of definiteness, that $i \leq j$. Again by (28) and (3),

$$\left|\mathfrak{F}\!\left(\left\{\frac{\gamma_l^2}{\lambda_l - z}\right\}_{l=1}^{i-1}\right)\mathfrak{F}\!\left(\left\{\frac{\gamma_l^2}{\lambda_l - z}\right\}_{l=j+1}^{\infty}\right)\mathfrak{F}\!\left(\left\{\frac{\gamma_l^2}{\lambda_l - z}\right\}_{l=1}^{\infty}\right)^{-1}\right| \leq C_1,$$

for all i, j. It remains to estimate the expression

$$\frac{1}{|\lambda_j - z|}\left|\prod_{l=i}^{j-1}\frac{w_l}{\lambda_l - z}\right|. \tag{38}$$

We distinguish three cases. For the finite set of couples i, j, $i \leq j \leq n_0$, (38) is bounded from above by a constant C_2. Using (37), if $i \leq n_0 \leq j$ then (38) is majorized by

$$C_2\left|\frac{\lambda_{n_0} - z}{\lambda_j - z}\prod_{l=n_0}^{j-1}\frac{w_l}{\lambda_l - z}\right| \;\leq\; C_2\tau^{-1/2}|\lambda_{n_0} - z|^{1/2}\,2^{-j+n_0}.$$

Similarly, if $n_0 \leq i \leq j$ then (38) is majorized by $\tau^{-1}2^{-j+i}$. From these partial upper bounds the estimate (36) readily follows.

From (36) one deduces that the matrix $R(z)$ represents a bounded operator on $\ell^2(\mathbb{N})$. In fact, one can write $R(z)$ as a countable sum,

$$R(z) = \sum_{s\in\mathbb{Z}} R(z;s), \tag{39}$$

where the matrix elements of the summands are $R(z;s)_{i,j} = R(z)_{i,j}$ if $i - j = s$ and $R(z;s)_{i,j} = 0$ otherwise. Thus $R(z;s)$ has nonvanishing elements on only one parallel to the diagonal and

$$\|R(z;s)\| = \sup\{|R(z)_{i,j}|;\ i - j = s\} \leq C\,2^{-|s|}.$$

Hence the series (39) converges in the operator norm. With some abuse of notation, we shall denote the corresponding bounded operator again by the symbol $R(z)$.

Further one observes that, on the level of formal matrix products,

$$(\mathfrak{J} - z)R(z) = R(z)(\mathfrak{J} - z) = I.$$

The both equalities are in fact equivalent to the countable system of equations (with $w_0 = 0$)

$$w_{k-1}G(z;i,k-1) + (\lambda_k - z)G(z;i,k) + w_k G(z;i,k+1) = \delta_{i,k},\ \ i,k \in \mathbb{N}.$$

This can be verified, in a straightforward manner, with the aid of the rule (4) or some of its particular cases (5) and (6). By inspection of the domains one then readily shows that the operators $J_{\max} - z$ and $R(z)$ are mutually inverse and so $z \in \rho(J_{\max})$.

Finally, suppose that $|\lambda_n| \to \infty$ as $n \to \infty$, and $z \in \mathbb{C} \setminus \mathfrak{Z}(\mathcal{J})$. It turns out that then the above estimates may be somewhat refined. In particular, (38) is majorized by

$$|\lambda_i - z|^{-1/2}|\lambda_j - z|^{-1/2}\, 2^{-j+i}$$

for $n_0 \leq i, j$. But this implies that $R(z;s)_{i,j} \to 0$ as $i, j \to \infty$, with $i - j = s$ being constant. It follows that the operators $R(z;s)$ are compact. Since the series (39) converges in the operator norm, $R(z)$ is compact as well. □

Corollary 13. *If condition (28) is fulfilled for at least one $z_0 \in \mathbb{C}_0^\lambda$ then*

$$\operatorname{spec}(J_{max}) \setminus \operatorname{der}(\lambda) = \operatorname{spec}_p(J_{max}) \setminus \operatorname{der}(\lambda) = \mathfrak{Z}(\mathcal{J}).$$

Proof. Propositions 11 and 12 respectively imply the inclusions

$$\mathfrak{Z}(\mathcal{J}) \subset \operatorname{spec}_p(J_{\max}) \setminus \operatorname{der}(\lambda),\ \ \operatorname{spec}(J_{\max}) \setminus \operatorname{der}(\lambda) \subset \mathfrak{Z}(\mathcal{J}).$$

This shows the equality. □

Theorem 14. *Suppose that the convergence condition (28) is fulfilled for at least one $z_0 \in \mathbb{C}_0^\lambda$ and the function $F_{\mathcal{J}}(z)$ does not vanish identically on $\mathbb{C}_0^\lambda$. Then $J_{min} = J_{max} =: J$ and*

$$\operatorname{spec}(J) \setminus \operatorname{der}(\lambda) = \operatorname{spec}_p(J) \setminus \operatorname{der}(\lambda) = \mathfrak{Z}(\mathcal{J}). \tag{40}$$

Suppose, in addition, that the set $\mathbb{C} \setminus \operatorname{der}(\lambda)$ is connected. Then $\operatorname{spec}(J) \setminus \operatorname{der}(\lambda)$ consists of simple eigenvalues which have no accumulation points in $\mathbb{C} \setminus \operatorname{der}(\lambda)$.

Proof. By the assumptions, $\mathbb{C} \setminus (\operatorname{der}(\lambda) \cup \mathfrak{Z}(\mathcal{J})) \neq \emptyset$. From Proposition 12 one infers that $\rho(J_{\max}) \neq \emptyset$. According to Theorem 10, one has $J_{\min} = J_{\max}$. Then (40) becomes a particular case of Corollary 13.

Let us assume that $\mathbb{C} \setminus \operatorname{der}(\lambda)$ is connected. Then the set $\mathbb{C}_0^\lambda$ is clearly connected as well. Suppose on contrary that the point spectrum of J has an accumulation point in $\mathbb{C} \setminus \operatorname{der}(\lambda)$. Then, by equality (40), the set of zeros of the analytic function $F_{\mathcal{J}}(z)$ has an accumulation point in $\mathbb{C} \setminus \operatorname{der}(\lambda)$. This accumulation point may happen to be a member λ_n of the sequence λ, but then one knows that $F_{\mathcal{J}}(z)$ has a pole of finite order at λ_n. In any case, taking into account that $\mathbb{C}_0^\lambda$ is connected one comes to the conclusion that $F_{\mathcal{J}}(z) = 0$ everywhere on $\mathbb{C}_0^\lambda$, a contradiction. □

Remark 15. Theorem 14 is derived under two assumptions:
(i) The convergence condition (28) is fulfilled for at least one $z_0 \in \mathbb{C}_0^\lambda$.
(ii) The function $F_{\mathcal{J}}(z)$ does not vanish identically on $\mathbb{C}_0^\lambda$.
But let us point out that assumption (ii) is automatically fulfilled if (i) is true and the range of the sequence λ is contained in a halfplane. This happens, for example, if the sequence λ is real or the sequence $\{\operatorname{Re}\lambda_n\}_{n=1}^\infty$ is semibounded. In fact, let us for definiteness consider the latter case and suppose that $\operatorname{Re}\lambda_n \geq c$, $\forall n \in \mathbb{N}$. Then $(-\infty, c) \subset \mathbb{C}_0^\lambda$ and $1/|\lambda_n - z|$ tends to 0 monotonically for all n as $z \to -\infty$. Similarly as in (3) one derives the estimate

$$|F_{\mathcal{J}}(z) - 1| \leq \exp\left(\sum_{n=1}^{\infty}\left|\frac{w_n^2}{(\lambda_n - z)(\lambda_{n+1} - z)}\right|\right) - 1.$$

It follows that $\lim_{z\to-\infty} F_{\mathcal{J}}(z) = 1$. Notice that in the real case, the function $F_{\mathcal{J}}(z)$ can identically vanish neither on the upper nor on the lower halfplane.

Corollary 16. *Let $\mathcal{J}$ be real and suppose that (28) is fulfilled for at least one $z_0 \in \mathbb{C}_0^\lambda$. Then $J_{min} = J_{max} = J$ is self-adjoint and* $\operatorname{spec}(J) \setminus \operatorname{der}(\lambda) = \mathfrak{Z}(\mathcal{J})$ *consists of simple real eigenvalues which have no accumulation points in* $\mathbb{R} \setminus \operatorname{der}(\lambda)$.

Proof. Some assumptions in Theorem 14 become superfluous if $\mathcal{J}$ is real. As observed in Remark 15, assuming the convergence condition the function $F_{\mathcal{J}}(z)$ cannot vanish identically on $\mathbb{C}_0^\lambda$. The operator J is self-adjoint and may have only real eigenvalues. The set $\mathbb{C} \setminus \operatorname{der}(\lambda)$ may happen to be disconnected only if the range of the sequence λ is dense in $\mathbb{R}$, i.e. $\operatorname{der}(\lambda) = \mathbb{R}$. But even then the conclusion of the theorem remains trivially true. □

Let us complete this analysis by a formula for the norms of the eigenvectors described in Proposition 11. In order to simplify the discussion we restrict ourselves to the domain $\mathbb{C}_0^\lambda$. Then instead of (32) one may write

$$\xi_k(z) = \left(\prod_{l=1}^{k}\frac{w_{l-1}}{z - \lambda_l}\right)\mathfrak{F}\left(\left\{\frac{\gamma_l^2}{\lambda_l - z}\right\}_{l=k+1}^{\infty}\right),\ z \in \mathbb{C}_0^\lambda,\ k \in \mathbb{Z}_+. \tag{41}$$

This is in fact nothing but the solution f_n from (21) restricted to nonnegative indices.

Proposition 17. *If $z \in \mathbb{C}_0^\lambda$ satisfies (28) then the functions $\xi_k(z)$, $k \in \mathbb{Z}_+$, defined in (41) fulfill*

$$\sum_{k=1}^{\infty}\xi_k(z)^2 = \xi_0'(z)\xi_1(z) - \xi_0(z)\xi_1'(z). \tag{42}$$

Particularly, if in addition $\mathcal{J}$ is real and $z \in \mathbb{R} \cap \mathbb{C}_0^\lambda$ is an eigenvalue of J then $\xi(z) = (\xi_k(z))_{k=1}^\infty$ is a corresponding eigenvector and

$$\|\xi(z)\|^2 = \xi_0'(z)\xi_1(z). \tag{43}$$

Proof. Put $\zeta_j^{(1)} = z - \lambda_j$, $\zeta_j^{(2)} = y - \lambda_j$, $j \in \mathbb{N}$, in equation (23), where $z, y \in \mathbb{C}_0^\lambda$. Then Proposition 6 is applicable to $f_j^{(1)} = \xi_j(z)$, $f_j^{(2)} = \xi_j(y)$, $j \in \mathbb{Z}_+$. Hence ($w_0 = 1$)

$$(z - y) \sum_{k=0}^{\infty} \xi_k(z)\xi_k(y) = \xi_1(z)\xi_0(y) - \xi_0(z)\xi_1(y).$$

Now the limit $y \to z$ can be treated in a routine way. ☐

Corollary 18. *Suppose $\mathcal{J}$ is real and let condition (28) be fulfilled for at least one $z_0 \in \mathbb{C}_0^\lambda$. Then the function $F_\mathcal{J}(z)$ has only simple real zeros on $\mathbb{C}_0^\lambda$.*

Proof. Suppose $z \in \mathbb{C}_0^\lambda$ is a zero of $F_\mathcal{J}(z)$, i.e. $F_\mathcal{J}(z) \equiv \xi_0(z) = 0$. Then z is a real eigenvalue of J where $J = J_{\max} = J_{\min}$ is self-adjoint, as we know from Corollary 16. Moreover, by Proposition 11, $\xi(z) \neq 0$ is a corresponding real eigenvector. Hence from (43) one infers that necessarily $\xi_0'(z) \neq 0$. ☐

3.4 Approximation of the spectrum by spectra of truncated matrices

Let us first introduce some additional notation which will be needed in the current subsection. W and L stand for the diagonal matrix operators whose diagonals equal w and λ, respectively. U designates the unilateral shift and U^* its adjoint operator ($Ue_n = e_{n+1}$ for $n = 1, 2, 3, \ldots$, with e_n being the vectors of the standard basis).

For a Jacobi matrix $\mathcal{J}$ we introduce the set

$$\Lambda(\mathcal{J}) := \{\mu \in \mathbb{C};\ \lim_{n\to\infty} \operatorname{dist}(\operatorname{spec}(J_n), \mu) = 0\} \tag{44}$$

where J_n is defined in (10) but now with $v = w$. Thus $\mu \in \Lambda(\mathcal{J})$ iff there exists a sequence of eigenvalues $\{\mu_n\}$ of J_n such that $\mu_n \to \mu$ as $n \to \infty$.

Remark 19. (i) Definition (44) is taken, for example, from [3]. Some authors prefer to work, however, with the set $\tilde{\Lambda}(\mathcal{J})$ formed by all limit points of sequences $\{\mu_k\}$ of eigenvalues from $\operatorname{spec}(J_{n_k})$, with $\{n_k\}$ being any possible strictly increasing sequence of indices [11, 12]. One clearly has $\Lambda(\mathcal{J}) \subset \tilde{\Lambda}(\mathcal{J})$ and the inclusion is in general strict as demonstrated by a simple example in [11, Eq. (2)].

(ii) In the case where the sequences λ and w are real and positive, respectively, the relation between the spectrum of a Jacobi operator J and the set $\Lambda(\mathcal{J})$ has been intensively studied by several authors. Most results of this kind are restricted to the bounded case, however. The inclusion $\operatorname{spec}(J) \subset \Lambda(\mathcal{J})$ is proved for bounded Jacobi matrices in [3, Theorem 2.3]. It cannot be replaced, in general, by the sign of equality as demonstrates an counterexample constructed in the Appendix in [3].

(iii) In [14, Theorem 5.1] it is shown that $\operatorname{spec}(J_S) \subset \Lambda(\mathcal{J})$ where the operator

$$J_S := L + WU^* + UW$$

(an operator sum) is assumed to be self-adjoint. Some reasoning used below, notably that in Lemmas 21 and 22, is inspired by this article. We also remark that Theorem 23 below partially reproduces and overlaps with Theorems 2.1 and 2.10 from [12].

Lemma 20. *If condition (28) is fulfilled for at least one $z_0 \in \mathbb{C}_0^\lambda$ then*

$$\Lambda(\mathcal{J}) \setminus \operatorname{der}(\lambda) \subset \mathfrak{Z}(\mathcal{J}). \tag{45}$$

Proof. Let $z \in \Lambda(\mathcal{J}) \setminus \operatorname{der}(\lambda)$. Then, by definition, there exists a sequence $\{\mu_n\}$ such that $\mu_n \in \operatorname{spec}(J_n)$ and $\mu_n \to z$ as $n \to \infty$. Considering sufficiently large indices n one may assume that μ_n does not coincide with any member λ_k of the sequence λ, possibly except of z. Using (13) one gets, for all sufficiently large n,

$$0 = \det(\mu_n I_n - J_n) = \left(\prod_{\substack{k=1 \\ \lambda_k \neq z}}^{n} (\mu_n - \lambda_k) \right) \lim_{u \to \mu_n} (u - z)^{r(z)} \, \mathfrak{F}\left(\left\{ \frac{\gamma_l^2}{\lambda_l - u} \right\}_{l=1}^{n} \right). \tag{46}$$

Denote temporarily

$$\tilde{F}^n(u) = (u - z)^{r(z)} \, \mathfrak{F}\left(\left\{ \frac{\gamma_l^2}{\lambda_l - u} \right\}_{l=1}^{n} \right), \; u \in \mathbb{C}_0^\lambda, \; n \in \mathbb{N} \cup \{\infty\}.$$

All functions $\tilde{F}^n(u)$ have a removable singularity at $u = z$. Moreover, with some slight modification of Lemma 8 one can show that $\tilde{F}^n(u) \to \tilde{F}^\infty(u)$ as $n \to \infty$ uniformly on a neighborhood of the point z. Since all the involved functions are analytic on this domain, one knows that the corresponding derivatives converge uniformly as well. Equation (46) means that $\tilde{F}^n(\mu_n) = 0$ for all n sufficiently large. One concludes that

$$\lim_{u \to z} (u - z)^{r(z)} \, F_{\mathcal{J}}(u) = \tilde{F}^\infty(z) = \lim_{n \to \infty} \tilde{F}^n(\mu_n) = 0.$$

Hence $z \in \mathfrak{Z}(\mathcal{J})$. □

Denote by P_n, $n \in \mathbb{N}$, the orthogonal projection on $\ell^2(\mathbb{N})$ onto the linear hull of the first n vectors of the standard basis. Then the finite-dimensional operator J_n introduced in (10) can be identified with $P_n J P_n$ restricted to the subspace $\operatorname{Ran} P_n$. Here J is any operator such that $\dot{J} \subset J$ (see Definition 9).

Lemma 21. *Let J be any operator on $\ell^2(\mathbb{N})$ fulfilling $\dot{J} \subset J$. Then (in $\ell^2(\mathbb{N})$)*

$$\lim_{n \to \infty} P_n J P_n x = Jx, \; \forall x \in \operatorname{Dom}(J) \cap \operatorname{Dom}(WU^*).$$

Proof. Let $x \in \operatorname{Dom}(J) \cap \operatorname{Dom}(WU^*)$. Since

$$(J - P_n J P_n)x = (I - P_n)Jx + P_n J (I - P_n)x$$

one has

$$\|(J - P_n J P_n)x\|^2 = \|(I - P_n)Jx\|^2 + |w_n|^2 |\langle x, e_{n+1} \rangle|^2 = \|(I - P_n)Jx\|^2 + |\langle WU^* x, e_n \rangle|^2.$$

The RHS clearly tends to zero as $n \to \infty$. □

Lemma 22. *Suppose that $\mathcal{J}$ is real and let J be a self-adjoint extension of the symmetric operator J_{min}. If* $\mathrm{Dom}(J) \subset \mathrm{Dom}(WU^*)$ *then* $\mathrm{spec}(J) \subset \Lambda(\mathcal{J})$*. If, in addition, condition (28) is fulfilled for at least one $z_0 \in \mathbb{C}_0^\lambda$ then*

$$\mathfrak{Z}(\mathcal{J}) = \mathrm{spec}(J) \setminus \mathrm{der}(\lambda) = \mathrm{spec}_p(J) \setminus \mathrm{der}(\lambda) = \Lambda(\mathcal{J}) \setminus \mathrm{der}(\lambda). \tag{47}$$

Proof. First suppose that, on the contrary, $\mu \in \mathrm{spec}(J)$ and $\mu \notin \Lambda(\mathcal{J})$. Then there exist $d > 0$ and a subsequence $\{J_{n_k}\}_{k=1}^{\infty}$ such that $\mathrm{dist}(\mu, \mathrm{spec}\, J_{n_k}) \geq d$ for all k. Let x be an arbitrary vector from $\mathrm{Dom}(J) \subset \mathrm{Dom}(WU^*)$.

As is well known and easy to verify, if A is a self-adjoint operator such that 0 belongs to its resolvent set then $\|A^{-1}\| \leq 1/\,\mathrm{dist}(0, \mathrm{spec}\, A)$. This implies that

$$\forall f \in \mathrm{Dom}\, A,\ \|Af\| \geq \mathrm{dist}(0, \mathrm{spec}\, A)\|f\|.$$

Applying this observation to the operators J_{n_k} one gets

$$\|(\mu P_{n_k} - P_{n_k} J P_{n_k})x\| \geq d\|P_{n_k} x\|.$$

Sending $k \to \infty$ and referring to Lemma 21 one concludes that $\|(\mu - J)x\| \geq d\|x\|$, for every $x \in \mathrm{Dom}(J)$. By the Weyl criterion $\mu \in \rho(J)$, a contradiction.

If, in addition, (28) is fulfilled for at least one $z_0 \in \mathbb{C}_0^\lambda$ then we know from Corollary 16 that J is in fact unique and the first two equalities in (47) are true. So combination of Lemma 20 with the first part of the current lemma yields

$$\mathfrak{Z}(\mathcal{J}) = \mathrm{spec}(J) \setminus \mathrm{der}(\lambda) \subset \Lambda(\mathcal{J}) \setminus \mathrm{der}(\lambda) \subset \mathfrak{Z}(\mathcal{J}).$$

This completes the proof. ☐

Theorem 23. *Suppose that the Jacobi matrix $\mathcal{J}$ is real and let condition (28) be fulfilled for at least one $z_0 \in \mathbb{C}_0^\lambda$. If any of the following two conditions is satisfied:*

(i) $\lim_{n\to\infty} w_n = 0$,

(ii) $\lim_{n\to\infty} |\lambda_n| = \infty$ *and*

$$\limsup_{n\to\infty} \frac{|w_n|}{|\lambda_n|} + \limsup_{n\to\infty} \frac{|w_n|}{|\lambda_{n+1}|} < 1, \tag{48}$$

then

$$\Lambda(\mathcal{J}) = \mathrm{spec}(J), \tag{49}$$

where J is the unique self-adjoint operator whose matrix in the standard basis equals $\mathcal{J}$.

Proof. According to Corollary 16, $J = J_{\min} = J_{\max}$ is the unique self-adjoint operator determined by $\mathcal{J}$.

(i) Let $\lim_{n\to\infty} w_n = 0$. Then we claim that

$$\mathrm{der}(\lambda) = \mathrm{spec}_{ess}(J) \subset \mathrm{spec}(J) \subset \Lambda(\mathcal{J}). \tag{50}$$

In fact, the Hermitian operator $WU^* + UW$ is compact and

$$J = L + (WU^* + UW) \tag{51}$$

(an operator sum, $\operatorname{Dom} J = \operatorname{Dom} L$). But the Weyl theorem tells us that J has the same essential spectrum as L, which is nothing but $\operatorname{der}(\lambda)$. The last inclusion in (50) follows from Lemma 22. Again by Lemma 22, $\operatorname{spec}(J) \setminus \operatorname{der}(\lambda) = \Lambda(\mathcal{J}) \setminus \operatorname{der}(\lambda)$, which jointly with (50) implies (49).

(ii) Suppose that $|\lambda_n| \to \infty$ and (48) is true. Then clearly $\operatorname{der}(\lambda) = \emptyset$. We wish to verify the assumptions of Lemma 22. Since the sum of a self-adjoint operator with a bounded Hermitian operator does not change the domain one may suppose, without loss of generality, that

$$|\lambda_n| \geq 1, \ \forall n, \ \text{ and } \ \sup_n \frac{|w_n|}{|\lambda_n|} + \sup_n \frac{|w_n|}{|\lambda_{n+1}|} \leq \alpha < 1,$$

where α is a constant. But this means that

$$\|(WU^* + UW)L^{-1}\| \leq \|WU^*L^{-1}\| + \|WL^{-1}\| \leq \alpha,$$

and hence $(WU^* + UW)$ is L-bounded with a relative bound smaller than 1. By the Kato-Rellich theorem [17, Theorem V.4.3], (51) is again true. Moreover,

$$\operatorname{Dom} J = \operatorname{Dom} L \subset \operatorname{Dom}(WU^*).$$

Hence in this case, too, Lemma 22 implies (49). □

Remark 24. Observe that (28) already implies that

$$\lim_{n\to\infty} \left| \frac{w_n^2}{\lambda_n \lambda_{n+1}} \right| = 0.$$

Thus assumption (48) may well turn out to be superfluous in some concrete examples.

4 Examples

4.1 Explicitly solvable examples of point spectra

In all examples presented below the Jacobi matrix $\mathcal{J}$ is real and symmetric. The set of accumulation points $\operatorname{der}(\lambda)$ is either empty or the one-point set $\{0\}$. Moreover, condition (28) is readily checked to be satisfied for any $z_0 \in \mathbb{C} \setminus \mathbb{R}$. Thus Corollary 16 applies to all these examples and may be used to determine the spectrum of the unique self-adjoint operator J determined by $\mathcal{J}$ (recall also definition (31) of the zero set of the characteristic function). In addition, Proposition 11 and equation (32) (or (41)) provide us with explicit formulas for the corresponding eigenvectors.

Example 25. This is an example of an unbounded Jacobi operator. Let $\lambda_n = n\alpha$, where $\alpha \in \mathbb{R} \setminus \{0\}$, and $w_n = w > 0$ for all $n \in \mathbb{N}$. Thus

$$J = \begin{pmatrix} \alpha & w & & & \\ w & 2\alpha & w & & \\ & w & 3\alpha & w & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

One has $\operatorname{der}(\lambda) = \emptyset$ and $\operatorname{spec}(J) = \mathfrak{Z}(\mathcal{J})$. Using (15) one derives that

$$\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\alpha k - z}\right\}_{k=r+1}^{\infty}\right) = \mathfrak{F}\left(\left\{\frac{w}{\alpha k - z}\right\}_{k=r+1}^{\infty}\right) = \left(\frac{w}{\alpha}\right)^{-r+z/\alpha} \Gamma\left(1 + r - \frac{z}{\alpha}\right) J_{r-z/\alpha}\left(\frac{2w}{\alpha}\right)$$

for $r \in \mathbb{Z}_+$. It follows that

$$\operatorname{spec}(J) = \left\{z \in \mathbb{C};\ J_{-z/\alpha}\left(\frac{2w}{\alpha}\right) = 0\right\}. \tag{52}$$

For the corresponding eigenvectors $v(z)$ one obtains

$$v_k(z) = (-1)^k J_{k-z/\alpha}\left(\frac{2w}{\alpha}\right),\ \ k \in \mathbb{N}.$$

Let us remark that the characterization of the spectrum of J, as given in (52), was observed earlier by several authors, see [15, Sec. 3] and [19, Thm. 3.1]. We discuss in more detail solutions of the characteristic equation $J_{-z}(2w) = 0$ below in Subsection 4.3.

Further we describe four examples in which the Jacobi matrix always represents a compact operator on $\ell^2(\mathbb{N})$. We shall make use of the following construction. Let us fix positive constants c, α and β. For $n \in \mathbb{Z}_+$ we define the c-deformed number n as

$$[n]_c = \sum_{i=0}^{n-1} c^i.$$

Hence $[n]_c = (c^n - 1)/(c - 1)$ if $c \neq 1$ and $[n]_c = n$ for $c = 1$. Notice that

$$\frac{[n+m-1]_c - [n-1]_c}{[m]_c} = [n]_c - [n-1]_c,\ \ \forall n, m \in \mathbb{N}. \tag{53}$$

As for the Jacobi matrix $\mathcal{J}$, we put

$$\lambda_n = \frac{1}{\alpha + [n-1]_c}\,,\ \ w_n = \beta\sqrt{\lambda_n - \lambda_{n+1}}\,, \quad n = 1, 2, 3, \ldots. \tag{54}$$

We claim that the identity

$$\sum_{k_1=r}^{\infty}\sum_{k_2=k_1+2}^{\infty}\cdots\sum_{k_s=k_{s-1}+2}^{\infty}$$
$$\times\frac{w_{k_1}^2}{(\lambda_{k_1}-z)(\lambda_{k_1+1}-z)}\frac{w_{k_2}^2}{(\lambda_{k_2}-z)(\lambda_{k_2+1}-z)}\cdots\frac{w_{k_s}^2}{(\lambda_{k_s}-z)(\lambda_{k_s+1}-z)}$$
$$=\frac{(-1)^s}{z^s}\,\beta^{2s}\prod_{i=1}^{s}\left([\,i\,]_c\left(1-\frac{z}{\lambda_{r+i-1}}\right)\right)^{-1}\tag{55}$$

holds for every $r,s\in\mathbb{N}$. In fact, to show (55) one can proceed by mathematical induction in s. The case $s=1$ as well as all induction steps are straightforward consequences of the equality

$$\frac{w_j^2}{(\lambda_j-z)(\lambda_{j+1}-z)}\prod_{i=1}^{s-1}\left([\,i\,]_c\left(1-\frac{z}{\lambda_{j+i+1}}\right)\right)^{-1}$$
$$=-\frac{\beta^2}{z}\left(\prod_{i=1}^{s}\left([\,i\,]_c\left(1-\frac{z}{\lambda_{j+i-1}}\right)\right)^{-1}-\prod_{i=1}^{s}\left([\,i\,]_c\left(1-\frac{z}{\lambda_{j+i}}\right)\right)^{-1}\right),$$

with $s=1,2,3,\ldots$, which in turn can be verified with the aid of (53).

A principal consequence one can draw from (55) is that

$$\mathfrak{F}\left(\left\{\frac{\gamma_n^2}{\lambda_n-z}\right\}_{n=r+1}^{\infty}\right)=\sum_{s=0}^{\infty}\frac{\beta^{2s}}{z^s}\prod_{i=1}^{s}\left([\,i\,]_c\left(1-\frac{z}{\lambda_{r+i}}\right)\right)^{-1}\tag{56}$$

holds for all $r\in\mathbb{Z}_+$.

Example 26. In (54), let us put $c=1$ and $\alpha=1$ while β is arbitrary positive. Then $\lambda_n=1/n$, $w_n=\beta/\sqrt{n(n+1)}$, for all $n\in\mathbb{N}$, and so

$$J=\begin{pmatrix}1&\beta/\sqrt{2}&&&\\ \beta/\sqrt{2}&1/2&\beta/\sqrt{6}&&\\ &\beta/\sqrt{6}&1/3&\beta/\sqrt{12}&\\ &&\ddots&\ddots&\ddots\end{pmatrix}.$$

One finds that

$$\begin{aligned}\mathfrak{F}\left(\left\{\frac{\gamma_n^2}{\lambda_n-z}\right\}_{n=k+1}^{\infty}\right)&=\sum_{s=0}^{\infty}\frac{\beta^{2s}}{s!\,z^s}\prod_{j=1}^{s}\frac{1}{1-(k+j)z}={}_0F_1\left(k+1-\frac{1}{z};-\frac{\beta^2}{z^2}\right)\\&=\left(\frac{z}{\beta}\right)^{k-1/z}\Gamma\left(k+1-\frac{1}{z}\right)J_{k-1/z}\left(\frac{2\beta}{z}\right),\end{aligned}\tag{57}$$

with $k\in\mathbb{Z}_+$, see [1, Eq. 9.1.69]. Then

$$F_{\mathcal{J}}(z)=\Gamma\left(1-\frac{1}{z}\right)\left(\frac{z}{\beta}\right)^{-1/z}J_{-1/z}\left(\frac{2\beta}{z}\right)$$

and

$$\operatorname{spec}(J)=\left\{z\in\mathbb{R}\setminus\{0\};\ J_{-1/z}\!\left(\frac{2\beta}{z}\right)=0\right\}\cup\{0\}.$$

For the corresponding eigenvectors $v(z)$ one has

$$v_k(z)=\sqrt{k}\,z^{-1/z}J_{k-1/z}\!\left(\frac{2\beta}{z}\right),\ k\in\mathbb{N}.$$

Example 27. Now we suppose in (54) that $c>1$ and put $\alpha=1/(c-1)$. Then $\lambda_n=(c-1)c^{-n+1}$ and $w_n=\beta\,(c-1)c^{-1/2}c^{(-n+1)/2}$. In order to simplify the expressions let us divide all matrix elements by the term $c-1$. Furthermore, we also replace the parameter β by $\beta c^{1/2}$ and use the substitution $c=1/q$, with $0<q<1$. Thus for the matrix simplified in this way we have $\lambda_n=q^{n-1}$ and $w_n=\beta q^{(n-1)/2}$. Hence

$$J=\begin{pmatrix}1 & \beta & & & \\ \beta & q & \beta\sqrt{q} & & \\ & \beta\sqrt{q} & q^2 & \beta q & \\ & & \ddots & \ddots & \ddots\end{pmatrix}.$$

Using (17), equation (56) then becomes

$$\begin{aligned}\mathfrak{F}\left(\left\{\frac{\gamma_n^2}{\lambda_n-z}\right\}_{n=r+1}^{\infty}\right) &= \sum_{s=0}^{\infty}(-1)^s\frac{q^{s(s-1)}}{(q;q)_s(q^r/z;q)_s}\left(\frac{q^r\beta}{z}\right)^{2s}\\ &= {}_0\phi_1(;q^r/z;q,-q^r\beta^2/z^2),\ r\in\mathbb{Z}_+.\end{aligned}$$

Thus we get

$$\operatorname{spec}(J)=\left\{z\in\mathbb{R}\setminus\{0\};\ \left(\frac{1}{z};q\right)_{\infty}\,{}_0\phi_1\!\left(;\frac{1}{z};q,-\frac{\beta^2}{z^2}\right)=0\right\}\cup\{0\}.$$

The kth entry of an eigenvector $v(z)$ corresponding to a nonzero point of the spectrum z may be written in the form

$$v_k(z)=q^{(k-1)(k-2)/4}\left(\frac{\beta}{z}\right)^{k-1}\left(\frac{q^k}{z};q\right)_{\infty}\,{}_0\phi_1\!\left(;\frac{q^k}{z};q,-\frac{q^k\beta^2}{z^2}\right),\ k\in\mathbb{N}.$$

Further we shortly discuss two examples of Jacobi matrices with zero diagonal and $w\in\ell^2(\mathbb{N})$. Such a Jacobi matrix represents a compact operator (even Hilbert-Schmidt). The characteristic function is an even function,

$$\mathfrak{F}\left(\left\{\frac{\gamma_n^2}{z}\right\}_{n=1}^{\infty}\right)=\sum_{m=0}^{\infty}\frac{(-1)^m}{z^{2m}}\sum_{k_1=1}^{\infty}\sum_{k_2=k_1+2}^{\infty}\cdots\sum_{k_m=k_{m-1}+2}^{\infty}w_{k_1}^2w_{k_2}^2\ldots w_{k_m}^2.$$

Hence the spectrum of J is symmetric with respect to the origin.

Though 0 always belongs to the spectrum of a compact Jacobi operator, one may ask under which conditions 0 is even an eigenvalue (necessarily simple). An answer

can be deduced directly from the eigenvalue equation (11). One immediately finds that any eigenvector x must satisfy $x_{2k} = 0$ and $x_{2k-1} = (-1)^{k+1}x_1/\gamma_{2k-1}$, $k \in \mathbb{N}$. Consequently, zero is a simple eigenvalue of J iff

$$\sum_{k=1}^{\infty} \frac{1}{\gamma_{2k-1}^2} < \infty. \tag{58}$$

Example 28. Let $\lambda_n = 0$ and $w_n = 1/\sqrt{(n+\alpha)(n+\alpha+1)}$, $n \in \mathbb{N}$, where $\alpha > -1$ is fixed. According to (15) one has, for $k \in \mathbb{Z}_+$,

$$\mathfrak{F}\left(\left\{\frac{\gamma_n^2}{z}\right\}_{n=k+1}^{\infty}\right) = \mathfrak{F}\left(\left\{\frac{1}{z(\alpha+n)}\right\}_{n=k+1}^{\infty}\right) = \Gamma(\alpha+k+1)z^{\alpha+k}J_{\alpha+k}\left(\frac{2}{z}\right).$$

Hence

$$\operatorname{spec}(J) = \left\{z \in \mathbb{R}\backslash\{0\};\ J_\alpha\left(\frac{2}{z}\right) = 0\right\} \cup \{0\}.$$

The kth entry of an eigenvector $v(z)$ corresponding to a nonzero eigenvalue z may be written in the form

$$v_k(z) = \sqrt{\alpha+k}\, z^\alpha J_{\alpha+k}\left(\frac{2}{z}\right),\ k \in \mathbb{N}.$$

It is well known that for $\alpha \in 1/2 + \mathbb{Z}$, the Bessel function $J_\alpha(z)$ can be expressed as a linear combination of sine and cosine functions, the simplest cases being

$$J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\cos(z),\ J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sin(z).$$

Thus for $\alpha = \pm 1/2$ the spectrum of J is described fully explicitly. In other cases the eigenvalues of J close to zero can approximately be determined from the known asymptotic formulas for large zeros of Bessel functions, see [1, Eq. 9.5.12].

Example 29. Suppose that $0 < q < 1$ and put $\lambda_n = 0$, $w_n = q^{n-1}$, $n \in \mathbb{N}$. With the aid of (17) one derives that

$$\mathfrak{F}\left(\left\{\frac{\gamma_n^2}{z}\right\}_{n=k+1}^{\infty}\right) = {}_0\phi_1(;0;q^2,-q^{2k}z^{-2}),\ k \in \mathbb{Z}_+.$$

It follows that

$$\operatorname{spec}(J) = \{z \in \mathbb{R} \setminus \{0\};\ {}_0\phi_1(;0;q^2,-z^{-2}) = 0\} \cup \{0\}.$$

The components of an eigenvector $v(z)$ corresponding to an eigenvalue $z \neq 0$ may be expressed as

$$v_k(z) = q^{(k-1)(k-2)/2}z^{-k+1}\,{}_0\phi_1(;0;q^2,-q^{2k}z^{-2}),\ k \in \mathbb{N}.$$

In this example as well as in the previous one, 0 belongs to the continuous spectrum of J since the condition (58) is not fulfilled.

4.2 Applications of Proposition 17

Here we apply identity (42) to the five examples of Jacobi matrices described above. Without going into details, the final form of the presented identities is achieved after some simple substitutions. On the other hand, no attempt is made here to optimize the range of involved parameters; it is basically the same as it was for the Jacobi matrix in question.

1) In case of Example 25 one gets

$$\sum_{k=1}^{\infty} J_{\nu+k}(x)^2 = \frac{x}{2}\left(J_\nu(x)\,\frac{\partial}{\partial\nu}J_{\nu+1}(x) - J_{\nu+1}(x)\,\frac{\partial}{\partial\nu}J_\nu(x)\right),$$

where $x>0$ and $\nu\in\mathbb{C}$. This is in fact a particular case of (18).

2) In case of Example 26 one gets

$$\sum_{k=1}^{\infty} k\,J_{-\alpha z+k}(z)^2 = \frac{z^2}{2}\left(J_{-\alpha z}(z)\,\frac{\mathrm{d}}{\mathrm{d}z}J_{-\alpha z+1}(z) - J_{-\alpha z+1}(z)\,\frac{\mathrm{d}}{\mathrm{d}z}J_{-\alpha z}(z)\right),$$

where $\alpha>0$ and $z\in\mathbb{C}$.

3) In case of Example 28 one gets

$$\sum_{k=1}^{\infty} (\alpha+k)J_{\alpha+k}(z)^2 = \frac{z^2}{2}\left(J_{\alpha}(z)\,\frac{\mathrm{d}}{\mathrm{d}z}J_{\alpha+1}(z) - J_{\alpha+1}(z)\,\frac{\mathrm{d}}{\mathrm{d}z}J_{\alpha}(z)\right),$$

where $\alpha>-1$ and $z\in\mathbb{C}$.

4) In case of Example 27 one gets

$$\begin{aligned}
&\sum_{k=1}^{\infty} q^{(k-1)(k-2)/2}\,(\beta z)^{2k-2}\left((q^k z;q)_\infty\ {}_0\phi_1(;q^k z;q,-q^k\beta^2z^2)\right)^2\\
&= (qz;q)_\infty^{\,2}\Bigg({}_0\phi_1(;z;q,-\beta^2z^2)\,{}_0\phi_1(;qz;q,-q\beta^2z^2)\\
&\qquad\qquad + z\,(z-1)\Big({}_0\phi_1(;qz;q,-q\beta^2z^2)\,\frac{\mathrm{d}}{\mathrm{d}z}\,{}_0\phi_1(;z;q,-\beta^2z^2)\\
&\qquad\qquad\qquad - {}_0\phi_1(;z;q,-\beta^2z^2)\,\frac{\mathrm{d}}{\mathrm{d}z}\,{}_0\phi_1(;qz;q,-q\beta^2z^2)\Big)\Bigg),
\end{aligned}$$

where $0<q<1$, $\beta>0$ and $z\in\mathbb{C}$.

5) In case of Example 29 one gets

$$\begin{aligned}
\sum_{k=1}^{\infty} q^{(k-1)(k-2)/2}\,z^{k-1}\,{}_0\phi_1(;0;q,-q^kz)^2 \ &=\ {}_0\phi_1(;0;q,-z)\,{}_0\phi_1(;0;q,-qz)\\
&+2z\left({}_0\phi_1(;0;q,-z)\,\frac{\mathrm{d}}{\mathrm{d}z}\,{}_0\phi_1(;0;q,-qz) - {}_0\phi_1(;0;q,-qz)\,\frac{\mathrm{d}}{\mathrm{d}z}\,{}_0\phi_1(;0;q,-z)\right),
\end{aligned}$$

where $0<q<1$ and $z\in\mathbb{C}$.

4.3 A Jacobi matrix with a linear diagonal and constant parallels

Here we discuss in somewhat more detail Example 25 concerned with a Jacobi matrix having a linear diagonal and constant parallels. For simplicity and with no loss of generality we put $\alpha = 1$. Our goal is to study how the spectrum of the Jacobi operator J depends on the real parameter w. We treat J as a linear operator-valued function, $J = J(w)$. One may write $J(w) = L + wT$ where L is the diagonal operator with the diagonal sequence $\lambda_n = n$, $\forall n \in \mathbb{N}$, and T has all units on the parallels neighboring to the diagonal and all zeros elsewhere. Notice that $\|T\| \leq 2$.

We know that $J(w)$ has, for all $w \in \mathbb{R}$, a semibounded simple discrete spectrum. Let us enumerate the eigenvalues in ascending order as $\lambda_s(w)$, $s \in \mathbb{N}$. From the standard perturbation theory one infers that all functions $\lambda_s(w)$ are real analytic, with $\lambda_s(0) = s$. Moreover, the functions $\lambda_s(w)$ are also known to be even and so we restrict w to the positive real half-axis. In Example 25 we learned that for every $w > 0$ fixed, the roots of the equation $J_{-z}(2w) = 0$ are exactly $\lambda_s(w)$, $s \in \mathbb{N}$. Several first eigenvalues $\lambda_s(w)$ as functions of w are depicted in Figure 1.

The problem of roots of a Bessel function depending on the order, with the argument being fixed, has a long history. Here we make use of some results derived in the classical paper [6]. Some numerical aspects of the problem are discussed in [15]. For comparatively recent results in this domain one may consult [19] and references therein.

In [6] it is shown that

$$\frac{d\lambda_s(w)}{dw} = -\left(2w\int_0^\infty K_0(4w\sinh(t))\exp\left(2\lambda_s(w)t\right)dt\right)^{-1}.$$

From this relation one immediately deduces a few basic qualitative properties of the spectrum of the Jacobi operator.

Proposition 30 (M. J. Coulomb)**.** *The spectrum $\{\lambda_s(w);\ s \in \mathbb{N}\}$ of the above introduced Jacobi operator, depending on the parameter $w \geq 0$, has the following properties.*

(i) For every $s \in \mathbb{N}$, the function $\lambda_s(w)$ is strictly decreasing.

(ii) If $r < s$ then $\lambda_r{}'(w) < \lambda_s{}'(w)$.

(iii) In particular, the distance between two neighboring eigenvalues $\lambda_{s+1}(w) - \lambda_s(w)$, $s \in \mathbb{N}$, increases with increasing w and is always greater than or equal to 1, with the equality only for $w = 0$.

Let us next check the asymptotic behavior of $\lambda_s(w)$ at infinity. The asymptotic expansion at infinity of the sth root $j_s(\nu)$ of the equation $J_\nu(x) = 0$, with ν being fixed, reads [1, Eq. 9.5.22]

$$j_s(\nu) = \nu - 2^{-1/3}a_s\nu^{1/3} + O\big(\nu^{-1/3}\big) \ \text{ as } \nu \to +\infty,$$

where a_s is the sth negative zero of the Airy function $\mathrm{Ai}(x)$. From here one deduces that

$$\lambda_s(w) = -2w - a_s w^{1/3} + O\left(w^{-1/3}\right) \ \text{ as } w \to +\infty.$$

Concerning the asymptotic behavior of $\lambda_s(w)$ at $w=0$, one may use the expression for the Bessel function as a power series and apply the substitution, $\lambda_s(w)=s-z(w)$, $s=1,2,3,\ldots$. The solution $z=z(w)$, with $z(0)=0$, is then defined implicitly near $w=0$ by the equation

$$\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\Gamma(m+1-s+z(w))}\,w^{2m}=0.$$

The computation is straightforward and based on the relation

$$\frac{1}{\Gamma(-m+z)}=(-1)^m m!\left(z-\psi^{(0)}(m+1)z^2\right)+O\!\left(z^3\right),\ m=0,1,2,3,\ldots,$$

where $\psi^{(0)}$ is the polygamma function. This way one derives that, as $w\to 0$,

$$\begin{aligned}\lambda_1(w) &= 1-w^2+\frac{1}{2}w^4+O\!\left(w^6\right), \qquad (59)\\ \lambda_s(w) &= s-\frac{1}{(s-1)!s!}\,w^{2s}+\frac{2s}{(s-1)(s-1)!(s+1)!}\,w^{2s+2}+O\!\left(w^{2s+4}\right),\quad \text{for } s\geq 2.\end{aligned}$$

The same asymptotic formulas, as given in (59), can also be derived using the standard perturbation theory [17, § II.2]. Alternatively, one may use equivalent formulas for coefficients of the perturbation series derived in [7, 8] which are perhaps more convenient for this particular example.

The distance of $s\in\mathbb{N}$ to the rest of the spectrum of of the diagonal operator L equals 1. The Kato-Rellich theorem tells us that there exists exactly one eigenvalue of $J(w)$ in the disk centered at s and with radius $1/2$ as long as $|w|<1/4$. The explicit expression for the leading term in (59) suggests, however, that the eigenvalue $\lambda_s(w)$ may stay close to s on a much larger interval at least for high orders s. It turns out that actually $\lambda_s(w)$ is well approximated by this leading asymptotic term on an interval $[0,\beta_s)$, with $\beta_s\sim s/e$ for $s\gg 1$. A precise formulation is given in Proposition 33 below.

Denote by $y_k(\nu)$ the kth root of the Bessel function $Y_\nu(z)$, $k\in\mathbb{N}$. Let us put

$$\beta_s:=\left(\frac{(s-1)!\,s!}{\pi}\right)^{1/(2s)},\quad s\in\mathbb{N}.$$

In order to avoid confusion with the usual notation for Bessel functions, the nth truncation of $J(w)$ is now denoted by a bold letter as $\boldsymbol{J}_n(w)$.

Lemma 31. *The following estimate holds true:*

$$\beta_s<\frac{1}{2}\,y_1\!\left(s-\frac{1}{2}\right),\ \forall s\in\mathbb{N}. \qquad (60)$$

Proof. One knows that $\nu<y_1(\nu)$, $\forall\nu\geq 0$ [1, Eq. 9.5.2], and in particular this is true for $\nu=s-1/2$, $s\in\mathbb{N}$. On the other hand, the sequence

$$\phi_s=\frac{\pi}{(s-1)!\,s!}\left(s-\frac{1}{2}\right)^{2s}2^{-2s}$$

is readily verified to be increasing, and $1 < \phi_4$. This shows (60) for all $s \geq 4$. The cases $s = 1, 2, 3$ may be checked numerically. $\square$

Lemma 32. *Denote by $\chi_n(w; z)$ the characteristic polynomial of the nth truncation $\boldsymbol{J}_n(w)$ of the Jacobi matrix $J(w)$. If $0 \leq w \leq \beta_s$ for some $s \in \mathbb{N}$ then $z = \lambda_s(w)$ solves the equation*

$$\chi_{2s-1}(w; z) - \frac{w J_{2s-z}(2w)}{J_{2s-1-z}(2w)} \chi_{2s-2}(w; z) = 0. \tag{61}$$

Proof. Let $\{e_k;\, k \in \mathbb{N}\}$ be the standard basis in $\ell^2(\mathbb{N})$. Let us split the Hilbert space into the orthogonal sum

$$\ell^2(\mathbb{N}) = \operatorname{span}\{e_k;\, 1 \leq k \leq 2s-1\} \oplus \overline{\operatorname{span}\{e_k;\, 2s \leq k\}}.$$

Then $J(w)$ splits correspondingly into four matrix blocks,

$$J(w) = \begin{pmatrix} A(w) & B(w) \\ C(w) & D(w) \end{pmatrix}.$$

Here $A(w) = \boldsymbol{J}_{2s-1}(w)$, $D(w) = J(w) + (2s-1)I$, the block $B(w)$ has just one nonzero element in the lower left corner and $C(w)$ is transposed to $B(w)$.

By the min max principle, the minimal eigenvalue of $D(w)$ is greater than or equal to $2s - 2w$. Since $\lambda_s(w) \leq s$ one can estimate

$$\min \operatorname{spec}(D(w)) - \lambda_s(w) = \lambda_1(w) - \lambda_s(w) + 2s - 1 \geq s - 2w.$$

We claim that $0 \leq w \leq \beta_s$ implies $\min \operatorname{spec}(D(w)) - \lambda_s(w) > 0$. This is obvious for $s = 1$. For $s \geq 2$, it suffices to show that $\beta_s < s/2$. This can be readily done by induction in s. Hence, under this assumption, $D(w) - z$ is invertible for $z = \lambda_s(w)$.

Solving the eigenvalue equation $J(w)\boldsymbol{v} = z\boldsymbol{v}$ one can write the eigenvector as a sum $\boldsymbol{v} = \boldsymbol{x} + \boldsymbol{y}$, in accordance with the above orthogonal decomposition. If $D(w) - z$ is invertible then the eigenvalue equation reduces to the finite-dimensional linear system

$$\big(A - z - B(D - z)^{-1}C\big)\,\boldsymbol{x} = 0. \tag{62}$$

One observes that $B(D - z)^{-1}C$ has all entries equal to zero except of the element in the lower right corner. Using (35) and (15) one finds that this nonzero entry equals

$$w J_{2s-z}(2w) J_{2s-1-z}(2w)^{-1}.$$

Equation (61) then immediately follows from (62). $\square$

Proposition 33. *For $s \in \mathbb{N}$ and $0 \leq w \leq \beta_s$, one has*

$$0 \leq s - \lambda_s(w) \leq \frac{1}{\pi} \arcsin\!\left(\frac{\pi w^{2s}}{(s-1)!s!}\right).$$

Proof. We start from Lemma 32 and equation (61). Let us recall from [20, Proposition 30] that

$$\det\left(\boldsymbol{J}_{2s-1}(w) - s - x\right) = (-1)^s x \sum_{k=0}^{s-1} \binom{2s-k-1}{k} w^{2k} \prod_{j=1}^{s-k-1} \left(j^2 - x^2\right).$$

Hence if $z \in \mathbb{R}$, $|z - s| \leq 1$, then

$$|\chi_{2s-1}(w; z)| \geq |z - s| \prod_{j=1}^{s-1} \left(j^2 - (z-s)^2\right). \tag{63}$$

Since $J_{-s+1/2}(x) = (-1)^s Y_{s-1/2}(x)$ it is true that for $2w = y_1(s - 1/2)$ one has $\lambda_s(w) = s - 1/2$. Because of monotonicity of $\lambda_s(w)$ one makes the following observation: if $2w \leq y_1(s - 1/2)$ then $s \geq \lambda_s(w) \geq s - 1/2$.

By Lemma 31, if $w \leq \beta_s$ then $2w \leq y_1(s - 1/2)$, and so the estimate (63) applies for $z = \lambda_s(w)$. Using also Proposition 4 to express $\chi_{2s-2}(w; z)$ one derives from (61) that

$$|\lambda - s| \leq w \left|\frac{J_{2s-\lambda}(2w)}{J_{2s-1-\lambda}(2w)}\right| \left|\frac{s-\lambda}{2s-1-\lambda}\,\mathfrak{F}\!\left(\frac{w}{1-\lambda}, \frac{w}{2-\lambda}, \ldots, \frac{w}{2s-2-\lambda}\right)\right| \tag{64}$$

where as well as in the remainder of the proof we write for short λ instead of $\lambda_s(w)$.

Starting from the equation

$$\mathfrak{F}\!\left(\frac{w}{1-\lambda}, \frac{w}{2-\lambda}, \frac{w}{3-\lambda}, \ldots\right) = 0, \quad \text{with } \lambda = \lambda_s(w),$$

and using (4), (15) one derives that, for all $k \in \mathbb{Z}_+$,

$$\left(\prod_{j=1}^{k}(j-\lambda)\right) \mathfrak{F}\!\left(\frac{w}{1-\lambda}, \frac{w}{2-\lambda}, \ldots, \frac{w}{k-\lambda}\right) = w^k\,\frac{J_{k+1-\lambda}(2w)}{J_{1-\lambda}(2w)}\,. \tag{65}$$

Combining (64) and (65) we get (knowing that $0 \leq s - \lambda \leq 1/2$ for $\lambda = \lambda_s(w)$)

$$s - \lambda \leq w^{2s-1} \left|\left(\prod_{j=1}^{s-1}(\lambda - j)\, \prod_{j=1}^{s-1}(j + s - \lambda)\right)^{-1} \frac{J_{2s-\lambda}(2w)}{J_{1-\lambda}(2w)}\right|.$$

But notice that, by expressing the sine function as an infinite product,

$$\prod_{j=1}^{s-1}(\lambda - j)\, \prod_{j=1}^{s-1}(j + s - \lambda) = ((s-1)!)^2\,\frac{\sin(\pi(s-\lambda))}{\pi(s-\lambda)} \left(\prod_{j=s}^{\infty}\left(1 - \frac{(s-\lambda)^2}{j^2}\right)\right)^{-1}.$$

Hence

$$\sin(\pi(s-\lambda)) \leq \pi\,\frac{w^{2s-1}}{((s-1)!)^2} \left|\frac{J_{2s-\lambda}(2w)}{J_{1-\lambda}(2w)}\right|.$$

From (26) one gets, while taking into account that $J_{-\lambda}(2w)=0$,

$$\sin(\pi\lambda)=\pi w J_\lambda(2w)J_{1-\lambda}(2w).$$

In addition, one knows that

$$|J_\nu(x)|\leq\frac{1}{\Gamma(\nu+1)}\left|\frac{x}{2}\right|^\nu$$

provided $\nu>-1/2$ and $x\in\mathbb{R}$ [1, Eq. 9.1.62]. Hence

$$\sin(\pi(s-\lambda))^2\leq\frac{\pi^2w^{4s}}{((s-1)!)^2\,\Gamma(2s+1-\lambda)\Gamma(\lambda+1)}\,.$$

Writing $\lambda=s-\zeta$, with $0\leq\zeta\leq 1/2$, one has

$$\frac{d}{d\zeta}\log\!\left(\frac{1}{\Gamma(s+\zeta+1)\Gamma(s-\zeta+1)}\right)=-\psi^{(0)}(s+\zeta+1)+\psi^{(0)}(s-\zeta+1)<0.$$

Thus we arrive at the estimate

$$\sin(\pi(s-\lambda))^2\leq\frac{\pi^2w^{4s}}{((s-1)!)^2\,(s!)^2}\,.$$

To complete the proof it suffices to notice that the assumption $w\leq\beta_s$ means nothing but $w^{2s}/\left((s-1)!s!\right)\leq 1$, and it also implies that $0\leq s-\lambda\leq 1/2$. ☐

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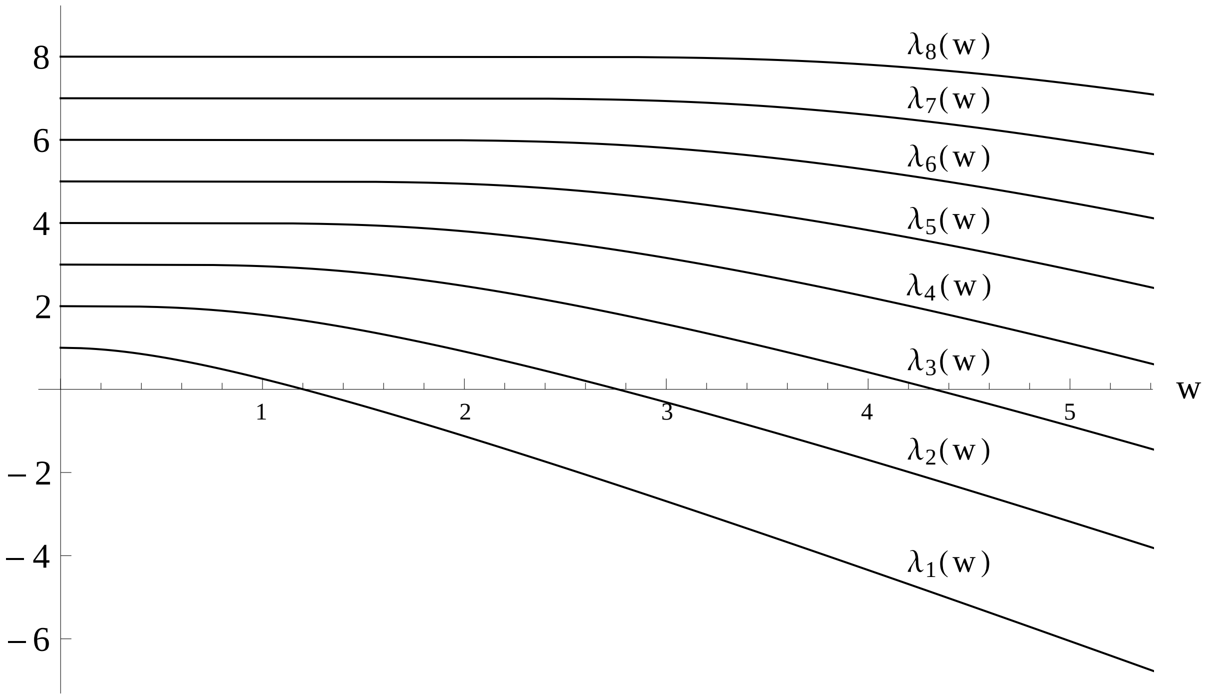

Figure 1. Several first eigenvalues $\lambda_s(w)$ as functions of the parameter w for the Jacobi operator $J = J(w)$ from Example 25, with $\alpha = 1$.